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Two More Classes of Games with the Continuous-time Fictitious Play Property

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Abstract

Fictitious Play is the oldest and most studied learning process for games. Since the already classical result for zero-sum games, convergence of beliefs to the set of Nash equilibria has been established for several classes of games, including weighted potential games, supermodular games with diminishing returns, and 3×3 supermodular games. Extending these results, we establish convergence of Continuous-time Fictitious Play for ordinal potential games and quasi-supermodular games with diminishing returns. As a by-product we obtain convergence for $3 \times m$ and 4×4 quasi-supermodular games. *JEL classification:* C72, D83.

Key words: Fictitious Play; Learning Process; Ordinal Potential Games; Quasi-Supermodular Games.

1 Introduction

The Fictitious Play Process was originally introduced by Brown (1949, 1951) as an algorithm to calculate the value of a two-person zero-sum game. Today, Fictitious Play serves as the prime example of myopic belief learning (see Fudenberg and Levine, 1998) and has its place in almost every modern textbook on game theory. Recently it has also proven useful as a simple optimization heuristic (Garcia et al., 2000, Lambert et al., 2005).

In this paper we focus on the continuous-time version of Fictitious Play. The belief paths of this process have been shown to converge to the set of Nash equilibria in several classes of games, among them two classes which are defined via cardinal conditions: supermodular games with diminishing returns and weighted potential games. We show here, that the defining cardinal conditions can be weakened considerably. Convergence indeed continues to hold under the

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respective ordinal conditions, i.e., in quasi-supermodular games and in ordinal potential games.

Brown (1949) introduced both the continuous-time and the discrete-time version of Fictitious Play. Due to its obvious shortcomings for computational purposes, the continuous-time version was neglected for many years until its revival by Rosenmüller (1971). However, for analytical studies the continuous-time version is often easier to handle than the discrete-time one, and mainly for this reason recent work on Fictitious Play has paid more attention to the continuous-time process. [1]

In a Discrete-time Fictitious Play (DFP) process two players are engaged in the repeated play of a bimatrix game. After an arbitrary initial move, in every round each player takes the empirical distribution of her opponent's strategies as her belief and responds with a pure strategy that maximizes her expected payoff, i.e., with a *myopic best response*. We say that a DFP process approaches equilibrium, if the sequence of beliefs converges to the set of Nash equilibria of the game. A game is said to have the *Discrete-time Fictitious Play property* (DFPP), if every DFP process approaches equilibrium in this game. Analogously, the continuous-time version of Fictitious Play is called Continuous-time Fictitious Play (CFP), and a game is said to have the *Continuous-time Fictitious Play Property* (CFPP), if every CFP process approaches equilibrium.

It is well known that there are games without the DFPP or the CFPP. [2] Shapley (1964) demonstrated this with an example of a 3×3 game where the beliefs of DFP converge to a limit cycle. The same phenomenon also occurs with CFP. Other classes of games where either DFP or CFP (or both) have been shown to fail convergence include Cowan (1992), Jordan (1993), Gaunersdorfer and Hofbauer (1995), Foster and Young (1998), and Krishna and Sjöström (1998).

However, most of the research concerned with Fictitious Play tried to identify classes of games where every Fictitious Play process approaches equilibrium. The largest classes of games where this is known (both for CFP and DFP) are zero-sum games (Robinson, 1951), $2 \times n$ games (Berger, 2005), dominance solvable games (Milgrom and Roberts, 1991), supermodular games with diminishing returns (Krishna, 1992) [3], and weighted potential games (Monderer

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[1] It should also be noted that this process is closely related to the *Best Response Dynamics* of Gilboa and Matsui (1991) and Matsui (1992), which some authors prefer to work with.

[2] Without assuming a particular tie-breaking rule, there are Fictitious Play processes that do not even approach equilibrium in 2×2 games, as shown by Monderer and Sela (1996). In such a case one needs a nondegeneracy condition, see Monderer and Shapley (1996). We therefore assume nondegenerate games throughout this paper.

[3] Krishna's 1992 working paper on DFP, though often cited, remained unpublished. Krishna and Sjöström (1997) present a proof for CFP, but refer back to Krishna

and Shapley, 1996). The modern proofs for the CFP versions are usually much easier than the older proofs for the DFP process,[4] (e.g. Rosenmüller, 1971, Hofbauer, 1995, or Harris, 1998). For overviews on these results see Krishna and Sjöström (1997) or Hofbauer and Sigmund (2003).

It is an open question if Fictitious Play also approaches equilibrium in supermodular games without diminishing returns. A small step in this direction was done by Hahn (1999), who showed that 3×3 supermodular games have the CFPP.

Milgrom and Shannon (1994) showed that many of the known results for supermodular games can already be derived under the weaker condition of *quasi*-supermodularity. Krishna (1992) raised the question, if quasi-supermodularity could already suffice for his result. However, as he explains, his proof does not extend to this larger class of games.

Monderer and Shapley (1996) defined the class of ordinal potential games, which contains the class of weighted potential games. For the latter, they prove convergence of beliefs to the equilibrium set. The conjecture that this result carries over from weighted to ordinal potential games has been explicitly doubted by Monderer and Sela (1997).

The present paper extends the described results of Krishna (1992), Hahn (1999), and Monderer and Shapley (1996) for the CFP process. We show that all nondegenerate *quasi*-supermodular games with diminishing returns have the CFPP. Concerning potential games, we prove that also nondegenerate games with an *ordinal* potential have the CFPP.[5] As a by-product, we finally obtain convergence of CFP in nondegenerate $3{\times}m$ and $4{\times}4$ quasi-supermodular games.

The remainder of this paper is structured as follows. In Section 2 we introduce the notation and terminology we use, and define supermodular and quasi-supermodular games, diminishing returns, nondegeneracy, weighted and ordinal potential games, Fictitious Play, and games with the pure Nash equilibrium property. Section 3 contains two important properties of CFP. In Section 4 we derive the first main result on ordinal potential games. Section 5 is concerned with quasi-supermodular games with diminishing returns, and contains the second main result. In Section 6 we turn to $3{\times}m$ and $4{\times}4$ quasi-supermodular games, and Section 7 concludes. The proofs of the last two results can be found in the Appendices A and B.

(1992) for a part of this proof.

[4] A three-line sketch of Robinson's (1951) celebrated result already appeared in Brown (1949) for the CFP process.

[5] Note that the set of ordinal potential games is much larger than the set of weighted potential games. The set of $n{\times}m$ bimatrix games can be identified with the Euclidean space $\mathbb{R}^{2nm}$. Within this space, the set of ordinal potential games contains an open set, while the set of weighted potential games has Lebesgue–measure zero (if $n \geq 2$ and $m \geq 3$).

2 Notation and Definitions

2.1 Bimatrix Games and Best Responses

Let (A, B) be a bimatrix game where player 1, the row player, has pure strategies $i \in N = \{1, 2, \ldots, n\}$, and player 2, the column player, has pure strategies $j \in M = \{1, 2, \ldots, m\}$. A and B are the $n \times m$ payoff matrices for players 1 and 2. Thus, if player 1 chooses $i \in N$ and player 2 chooses $j \in M$, the payoffs to players 1 and 2 are a_{ij} and b_{ij}, respectively. The set of mixed strategies of player 1 is the $n-1$ dimensional probability simplex S_n, and analogously S_m is the set of mixed strategies of player 2. With a little abuse of notation we will not distinguish between a pure strategy i of player 1 and the corresponding mixed strategy representation as the i-th unit vector $\mathbf{e}_i \in S_n$. Analogously we identify player 2's pure strategy j with the j-th unit vector $\mathbf{f}_j \in S_m$. Sometimes we will also speak of the players choosing a row, or column, respectively, of the bimatrix.

The expected payoff for player 1 playing strategy i against player 2's mixed strategy $\mathbf{y} = (y_1, \ldots, y_m)^t \in S_m$ (where the superscript t denotes the transpose of a vector or matrix) is $(A\mathbf{y})_i$. Analogously $(B^t\mathbf{x})_j$ is the expected payoff for player 2 playing strategy j against the mixed strategy $\mathbf{x} = (x_1, \ldots, x_n)^t \in S_n$. If both players use mixed strategies $\mathbf{x}$ and $\mathbf{y}$, respectively, the expected payoffs are $\mathbf{x} \cdot A\mathbf{y}$ to player 1 and $\mathbf{y} \cdot B^t\mathbf{x}$ to player 2, where the dot denotes the scalar product of two vectors. We denote by $BR_2(\mathbf{x})$ player 2's pure strategy best response correspondence, and by $br_2(\mathbf{x})$ her mixed strategy best response correspondence. Analogously, $BR_1(\mathbf{y})$ and $br_1(\mathbf{y})$ are the sets of player 1's pure and mixed best responses, respectively, to $\mathbf{y} \in S_m$. Let $BR(\mathbf{x}, \mathbf{y}) = BR_1(\mathbf{y}) \times BR_2(\mathbf{x})$ and $br(\mathbf{x}, \mathbf{y}) = br_1(\mathbf{y}) \times br_2(\mathbf{x})$. We say that $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is a best response to $(\mathbf{x}, \mathbf{y}) \in S_n \times S_m$, if $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in br(\mathbf{x}, \mathbf{y})$. Also, we call $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ a pure best response to $(\mathbf{x}, \mathbf{y})$, if $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in BR(\mathbf{x}, \mathbf{y})$. A strategy profile $(\mathbf{x}^*, \mathbf{y}^*)$ is a Nash equilibrium if and only if $(\mathbf{x}^*, \mathbf{y}^*) \in br(\mathbf{x}^*, \mathbf{y}^*)$. It is called a *pure* Nash equilibrium, if $(\mathbf{x}^*, \mathbf{y}^*) \in BR(\mathbf{x}^*, \mathbf{y}^*)$.

2.2 Quasi-Supermodular Games and Supermodular Games

Definition 1 *(i) A bimatrix game (A, B) is* quasi-supermodular*, if for all $i < i'$ and $j < j'$:*

$$a_{i'j} > a_{ij} \Longrightarrow a_{i'j'} > a_{ij'} \quad \text{and} \quad b_{ij'} > b_{ij} \Longrightarrow b_{i'j'} > b_{i'j}.$$

(ii) A bimatrix game (A, B) is supermodular*, if for all $i < i'$ and $j < j'$:*

$$a_{i'j'} - a_{ij'} > a_{i'j} - a_{ij} \quad \text{and} \quad b_{i'j'} - b_{i'j} > b_{ij'} - b_{ij}.$$

We write QSMG short for *quasi-supermodular game*, and SMG for *supermodular game.* In a QSMG, the payoff difference between two payoffs in a column of A or a line of B can change its sign at most once, and only from -1 to $+1$, if the players move up to a higher column or line, respectively. In a broader context, these games have been studied by Milgrom and Shannon (1994). From Definition 1, quasi-supermodularity is implied by supermodularity. In an SMG, the advantage of switching to a higher strategy increases when the opponent chooses a higher strategy.

Both QSMGs and SMGs have frequently been called games with *strategic complementarities*. We refrain from using this term to avoid confusion. Originally, the term "strategic complementarities" was coined by Bulow et al. (1985) to denote games with increasing best response correspondences. This property is already implied by quasi-supermodularity. SMGs have been introduced (in a much more general framework) by Topkis (1979) and studied by Vives (1990) and Milgrom and Roberts (1990). This class of games has important applications in economics, e.g. in models of oligopolistic competition, R&D competition, macroeconomic coordination, bank runs, network externalities, and many others.

2.3 Games with Diminishing Returns

Another condition we use is *diminishing returns*, sometimes also referred to as *diminishing marginal returns*. As the name suggests, this property means that the payoff advantage of increasing one's strategy is decreasing.

Definition 2 *A bimatrix game* (A, B) *has* diminishing returns *(DR), if for all* $i = 2, \dots, n-1$ *and* $j \in M$*,*

$$a_{i+1,j} - a_{ij} < a_{ij} - a_{i-1,j},$$

and for all $i \in N$ *and* $j = 2, \dots, m-1$*,*

$$b_{i,j+1} - b_{ij} < b_{ij} - b_{i,j-1}.$$

Krishna (1992) observed that DR restrict the best response correspondence in the following way.

Lemma 3 *Let* (A, B) *be a game with DR. Then for any* $(\mathbf{x}, \mathbf{y}) \in S_n \times S_m$*, the sets* $BR_1(\mathbf{y})$ *and* $BR_2(\mathbf{x})$ *contain at most two strategies. If one of these sets contains two strategies, they are numbered consecutively.*

2.4 Nondegenerate Games

As mentioned above, without assuming a tie-breaking rule, one must impose a nondegeneracy assumption in order to keep the CFPP, even in the class of 2×2 games. We work with games which are nondegenerate in the following specific sense.

Definition 4 *We call a bimatrix game* (A, B) degenerate*, if for some* $i, i' \in N$*, with* $i \neq i'$*, there exists* $j \in M$ *with* $a_{i'j} = a_{ij}$*, or if for some* $j, j' \in M$*, with* $j \neq j'$*, there exists* $i \in N$ *with* $b_{ij'} = b_{ij}$*. Otherwise, the game is said to be* nondegenerate*.*

We write ND((Q)SM)G short for *nondegenerate ((quasi-)supermodular) game.*

2.5 Potential Games

Monderer and Shapley (1996) define several classes of games with a so-called *potential.* The class of *ordinal potential games* contains the class of *weighted potential games.*

Definition 5 *(i) A bimatrix game* (A, B) *is an* ordinal potential game*, if there exists an ordinal potential function, i.e. a function* $F : N \times M \to \mathbb{R}$*, such that for all* $i, i' \in N$ *and* $j, j' \in M$*,*

$$a_{i'j} - a_{ij} > 0 \Leftrightarrow F(i', j) - F(i, j) > 0 \text{ and } b_{ij'} - b_{ij} > 0 \Leftrightarrow F(i, j') - F(i, j) > 0.$$

(ii) A bimatrix game (A, B) *is a* weighted potential game*, if there exist positive weights* w_1 *and* w_2 *and a function* $F : N \times M \to \mathbb{R}$*, such that for all* $i, i' \in N$ *and* $j, j' \in M$*,*

$$a_{i'j} - a_{ij} = w_1[F(i', j) - F(i, j)] \text{ and } b_{ij'} - b_{ij} = w_2[F(i, j') - F(i, j)].$$

They also define *improvement paths* and games with the *finite improvement property.* We extend this definition slightly by defining *improvement steps.*

Definition 6 *For a bimatrix game* (A, B)*, define the following binary relation on* $N \times M$*:* $(i, j) \to (i', j') \Leftrightarrow (i = i' \text{ and } b_{ij'} > b_{ij}) \text{ or } (j = j' \text{ and } a_{i'j} > a_{ij})$*. If* $(i, j) \to (i', j')$*, we say that this is an* improvement step*. We denote by* $|i' - i| + |j' - j|$ *the* length *of the improvement step. An* improvement path *is a (finite or infinite) sequence of improvement steps* $(i_1, j_1) \to (i_2, j_2) \to \cdots$ *in* $N \times M$*. An improvement path* $(i_1, j_1) \to \cdots \to (i_k, j_k)$ *is called an* improvement cycle*, if* $(i_k, j_k) = (i_1, j_1)$*. A bimatrix game is said to have the* finite improvement property *(FIP), if every improvement path is finite, i.e., if there are no improvement cycles.*

It is clear that every nondegenerate ordinal potential game (NDOPG) has the FIP. Monderer and Shapley (1996) show that also the opposite direction holds.

Lemma 7 *A nondegenerate bimatrix game has the FIP if and only if it is an ordinal potential game.*

Definition 8 *For the* $n \times m$ *bimatrix game* (A, B)*, the sequence* $(i_t, j_t)_{t\in\mathbb{N}}$ *is a* Discrete-time Fictitious Play process*, if* $(i_1, j_1) \in N \times M$ *and for all* $t \in \mathbb{N}$*,*

$$(i_{t+1}, j_{t+1}) \in BR(\mathbf{x}(t), \mathbf{y}(t)),$$

where the beliefs $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$ *are given by*

$$\mathbf{x}(t) = \frac{1}{t}\sum_{s=1}^{t} i_s \;\; \textit{and} \;\; \mathbf{y}(t) = \frac{1}{t}\sum_{s=1}^{t} j_s.$$ [6]

Note that the beliefs can be updated recursively. The belief of a player in round $t+1$ is a convex combination of his belief in round t and his opponent's move in round $t+1$:

$$(\mathbf{x}, \mathbf{y})(t+1) = \frac{t}{t+1}(\mathbf{x}, \mathbf{y})(t) + \frac{1}{t+1}(i_{t+1}, j_{t+1}). \tag{1}$$

Continuous-time Fictitious Play can roughly be described as the limiting "zero step size" version of DFP. Replacing the 1's in (1) by ϵ and letting $\epsilon \to 0$ we obtain CFP.

Definition 9 *A* strategy path *in* $N \times M$ *is a function* $[0, \infty] \to N \times M$, $t \mapsto (i_t, j_t)$*, whose points of discontinuity have no finite accumulation point in* $\mathbb{R}$*. For the* $n \times m$ *bimatrix game* (A, B)*, a strategy path* $(i_t, j_t)_{t\geq 0}$ *is a* Continuous-time Fictitious Play path*, if for almost all* $t \geq 1$*,*

$$(i_t, j_t) \in BR(\mathbf{x}(t), \mathbf{y}(t)),$$

where for $t > 0$ *the* belief path $(\mathbf{x}(t), \mathbf{y}(t))$ *is given by*

$$(\mathbf{x}, \mathbf{y})(t) = \frac{1}{t}\int_0^t (i_s, j_s)\, \mathrm{d}s.$$

If a (discrete- or continuous-time) Fictitious Play process converges, it must be constant from some stage on, implying convergence to the respective pure Nash equilibrium. Even if a process does not converge, it is easily established that if the beliefs converge, then the limit must be a Nash equilibrium. As noted above, however, there are games where the beliefs need not converge.

If at some point in time the best response set is multivalued, there may be several possible continuations of a Fictitious Play process. To handle this multiplicity of solutions, particular tie-breaking rules have sometimes been imposed.

[6] Brown's (1951) original definition has players updating alternatingly instead of simultaneously, but this version has disappeared from the literature, see Berger (2006).

Krishna (1992) or Hahn (1999) e.g. assumed that both players, whenever indifferent between two or more pure strategies, choose the strategy with the highest number. However, we do not impose any tie-breaking rule.

CFP is closely related to the *Best Response Dynamics*,

$$(\mathbf{x}, \mathbf{y})(1) \in S_n \times S_m, \quad (\dot{\mathbf{x}}, \dot{\mathbf{y}})(t) \in br((\mathbf{x}, \mathbf{y})(t)) - (\mathbf{x}, \mathbf{y})(t) \tag{2}$$

for almost all $t \geq 1$, a social learning process for large populations which was introduced by Gilboa and Matsui (1991) and Matsui (1992).[7]

It should be noted here, that—apart from tie-breaking rules—basically two different definitions of CFP can be found in the literature.[8] Definition 9 above is the one most commonly used. It implies that CFP belief paths are piecewise linear, pointing at pure-strategy pairs in the state space. As a consequence, CFP paths need not exist for all beliefs in all games. To see this, think of a Matching Pennies game where at time $t = 1$ the beliefs are at the mixed equilibrium. No CFP continuation of this path is possible. However, for all other beliefs at time $t = 1$, a CFP path exists. On the other hand, if the beliefs at $t = 1$ are at the completely mixed equilibrium of a pure coordination game, then there are as many CFP continuations as there are pure coordination equilibria, each belief path leading straight from the completely mixed equilibrium to one of these pure ones.

Harris (1998) takes a different approach and defines a Continuous-time Fictitious Play process as a solution—up to a rescaling of time—to the differential inclusion (2). Any CFP belief path according to Definition 9 is such a solution, but in general there are more solutions to (2) than there are CFP belief paths. For example, in both the Matching Pennies game and the pure coordination game depicted above, there are solutions to (2) which stay constant at the completely mixed equilibrium. The Best Response Dynamics has the advantage that solutions can be shown to exist through all initial values. However, in this paper we follow the majority and stick to the usual definition of CFP.

The analysis of Hofbauer (1995) shows that CFP paths exist in every game with the *pure Nash equilibrium property*. To explain this property, let $N' \subset N$ and $M' \subset M$ be nonempty subsets of the players' pure strategy sets. We call the restriction of the original game to these pure strategy subsets a *bimatrix-subgame*.[9] The following definition is from Takahashi and Yamamori (2002).

Definition 10 *A game has the* pure Nash equilibrium property *(PNEP), if every bimatrix-subgame has a pure Nash equilibrium.*

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[7] For an interesting connection between the Best Response Dynamics and Discrete-time Fictitious Play see Benaim et al. (2005).

[8] Brown's (1949) definition is informal, but corresponds to the definition we use here.

[9] Takahashi and Yamamori (2002) simply call it a *subgame*. To avoid confusion with the notion of a subgame in the context of extensive form games, we refrain from using this term.

Existence of CFP paths in games with the PNEP follows, because for every $(\mathbf{x}, \mathbf{y})(t) \in S_n \times S_m$, the CFP path can be continued with a pure Nash equilibrium of the bimatrix-subgame with $N' \times M' = BR(\mathbf{x}(t), \mathbf{y}(t))$.

Ordinal potential games, quasi-supermodular games, and dominance-solvable games have pure Nash equilibria. Since for games in these classes, any bimatrix-subgame is again a game of the same class, these games have the PNEP. The belief paths follow straight lines in the simplex product as long as the CFP path is constant. Whenever one of the players switches to another pure best response, the belief path changes its direction discontinuously. This can only happen when the belief path crosses a so-called *indifference hyperplane*, the set of mixed strategy profiles where one of the players is indifferent between two best responses. A formal definition of switching follows.

Definition 11 *We say that a CFP path (i_t, j_t)* switches *from (i, j) to (i', j') at time $t_1 > 1$, if $(i', j') \neq (i, j)$ and there exists $\epsilon > 0$ with $(i_t, j_t) = (i, j)$ for $t \in [t_1 - \epsilon, t_1[$ and $(i_t, j_t) = (i', j')$ for $t \in]t_1, t_1 + \epsilon]$.*

3 Two Improvement Principles for CFP

Whenever switching occurs along a CFP path, at least one of the players changes her strategy. The next lemma, called the *Improvement Principle*, is due to Monderer and Sela (1997), see also Sela (2000). It shows that the 'new' strategy of the switching player must be a weakly *better* response than her 'old' strategy against the 'old' strategy of the opponent. To keep the analysis self-contained, we repeat the proof here.

Lemma 12 *If a CFP path for the bimatrix game (A, B) switches from (i, j) to (i', j'), then*

$$a_{i'j} \geq a_{ij} \quad \text{and} \quad b_{ij'} \geq b_{ij}.$$

Proof. Let t_1 be the time where the process switches from (i, j) to (i', j'). Then both players are indifferent between their respective best responses, $\{(i, j), (i', j')\} \subset BR(\mathbf{x}(t_1), \mathbf{y}(t_1))$. By Definition 11 there exists $\epsilon > 0$ with $(i_t, j_t) = (i, j)$ for $t \in [t_0, t_1[$, where $t_0 = t_1 - \epsilon$. Hence $(\mathbf{x}(t_1), \mathbf{y}(t_1))$ is a convex combination of $(\mathbf{x}(t_0), \mathbf{y}(t_0))$ and (i, j), and we can write $(\mathbf{e}_i, \mathbf{f}_j) = c(\mathbf{x}(t_1), \mathbf{y}(t_1)) + (1-c)(\mathbf{x}(t_0), \mathbf{y}(t_0))$ for some $c \geq 1$. Left-multiplying the second component vector with the payoff matrix A yields $A\mathbf{f}_j = cA\mathbf{y}(t_1) + (1 - c)A\mathbf{y}(t_0)$. Subtracting the i-th line of this vector equation from the i'-th line gives

$$a_{i'j} - a_{ij} = c[(A\mathbf{y}(t_1))_{i'} - (A\mathbf{y}(t_1))_i] + (1 - c)[(A\mathbf{y}(t_0))_{i'} - (A\mathbf{y}(t_0))_i].$$

The first term on the right-hand side of this equation is zero, and the second term is nonnegative, since $i \in BR_1(\mathbf{y}(t_0))$ and $c \geq 1$. Hence $a_{i'j} \geq a_{ij}$. By the same reasoning we get $b_{ij'} \geq b_{ij}$. $\square$

The following complement of Lemma 12 is an essential requirement of our proofs. We call it the *Second Improvement Priciple.*

Lemma 13 *If a CFP process for the bimatrix game* (A, B) *switches from* (i, j) *to* (i', j')*, then*

$$a_{i'j'} \geq a_{ij'} \quad \textit{and} \quad b_{i'j'} \geq b_{i'j}.$$

Proof. Again let t_1 be the time where the process switches from (i, j) to (i', j'), then $\{(i, j), (i', j')\} \subset BR(\mathbf{x}(t_1), \mathbf{y}(t_1))$. By Definition 11 there exists $\epsilon > 0$ with $(i_t, j_t) = (i', j')$ for $t \in]t_1, t_2]$, where $t_2 = t_1 + \epsilon$. Hence $(\mathbf{x}(t_2), \mathbf{y}(t_2))$ is a convex combination of $(\mathbf{x}(t_1), \mathbf{y}(t_1))$ and (i', j'), and we can write $(\mathbf{e}_{i'}, \mathbf{f}_{j'}) = c(\mathbf{x}(t_2), \mathbf{y}(t_2)) + (1-c)(\mathbf{x}(t_1), \mathbf{y}(t_1))$ for some $c \geq 1$. Left-multiplying the second component vector with the payoff matrix A yields $A\mathbf{f}_{j'} = cA\mathbf{y}(t_2) + (1 - c)A\mathbf{y}(t_1)$. Subtracting the i-th line of this vector equation from the i'-th line gives

$$a_{i'j'} - a_{ij'} = c[(A\mathbf{y}(t_2))_{i'} - (A\mathbf{y}(t_2))_i] + (1 - c)[(A\mathbf{y}(t_1))_{i'} - (A\mathbf{y}(t_1))_i].$$

The second term on the right-hand side of this equation is zero, and the first term is nonnegative, since $i' \in BR_1(\mathbf{y}(t_2))$. Hence $a_{i'j'} \geq a_{ij'}$. By the same reasoning we get $b_{i'j'} \geq b_{i'j}$. □

If only one of the players switches her strategy at time t_1, i.e. if $i = i'$ or $j = j'$, then in both of these two Lemmas, one of the inequalities is trivially true, while the other inequalities are then identical. Only if both players switch simultaneously, Lemma 13 comes into use. If, moreover, the game is nondegenerate, then the strict inequalities hold. In this case we know that $(i, j) \to (i', j) \to (i', j')$, and also $(i, j) \to (i, j') \to (i', j')$.

4 CFP in Ordinal Potential Games

By the last remark, if a CFP path switches from (i, j) to (i', j') in a nondegenerate game, then these two pure strategy pairs are connected by an improvement path, even if both players switch simultaneously.

Lemma 14 *If a CFP path for the NDG* (A, B) *switches from* (i, j) *to* (i', j')*, then there is an improvement path from* (i, j) *to* (i', j')*.*

This is essentially all we need for the first main result.

Theorem 15 *Let* (A, B) *be an NDOPG. Then it has the CFPP.*

Proof. Assume there is an NDOPG without the CFPP. In a nonconvergent CFP path there are infinitely many switches. Since there are only finitely many pure-strategy pairs, however, the CFP path switches to one such pair infinitely often. Hence there must be a cyclic sequence of switches. By Lemma 14, this means that there is an improvement cycle. By Lemma 7 then, the game cannot have an ordinal potential, which contradicts the assumption. □

Fig. 1. The construction in the proof of Lemma 18. No sequence of length-1-improvement steps can return to the starting point (i^*, j^*).

5 CFP in Quasi-Supermodular Games with Diminishing Returns

If a CFP path switches from (i, j) to (i', j'), then at the switching point $(\mathbf{x}(t_1), \mathbf{y}(t_1))$, player 1 is indifferent between i and i', and player 2 is indifferent between j and j'. From Lemma 3, an immediate consequence of this is, that in games with DR, CFP can only switch to neighboring strategies.

Lemma 16 *Let (A, B) be a game with DR. If a CFP path switches from (i, j) to (i', j'), then $|i' - i| \leq 1$ and $|j' - j| \leq 1$.*

As a consequence, in an NDG with DR, any CFP path generates an improvement path where every improvement step has length 1. Lemma 14 then implies the following.

Lemma 17 *Let (A, B) be an NDG with DR. If a CFP path does not converge, then there exists an improvement cycle consisting of improvement steps of length 1.*

However, the next result states that in NDGs, quasi-supermodularity prevents the existence of such improvement cycles.

Lemma 18 *In an NDQSMG, every improvement path consisting of length-1-steps is finite.*

Proof. Let (A, B) be an NDQSMG. Assume the game admits an improvement cycle consisting of length-1-steps. Take any pair (i^*, j^*) in this cycle, where the next step is to $(i^*, j^* + 1)$—it is easy to see that such a pair always exists. By quasi-supermodularity, we have $(i', j^*) \to (i', j^* + 1)$ for all $i' \geq i^*$, see Figure 1. Since the cycle eventually returns to column j^*, there must be a line i^- with $(i^-, j^*+1) \to (i^-, j^*)$. We know that $i^- < i^*$ then. Since the improvement cycle leads from $(i^*, j^* + 1)$ to $(i^-, j^* + 1)$, it contains a step $(i^*, j^+) \to (i^* - 1, j^+)$ with $j^+ > j^*$. Then quasi-supermodularity implies $(i^*, j') \to (i^* - 1, j')$ for all $j' < j^+$, including all $j' \leq j^*$. But this implies that no improvement step of length 1 can enter the region of pairs (i, j) with $i \geq i^*$ and $j \leq j^*$ (coloured grey in Figure 1). Hence no sequence of such improvement steps can lead back from $(i^*, j^* + 1)$ to (i^*, j^*), contradicting the initial assumption. □

As a corollary of Lemmas 17 and 18, we obtain our second main result.

Theorem 19 *Every NDQSMG with DR has the CFPP.*

6 CFP in $3\times m$ and 4×4 Quasi-Supermodular Games

It has been conjectured (e.g. Krishna and Sjöström, 1997) that in (nondegenerate) SMGs, the condition of DR is not really necessary for CFP to converge. Hahn (1999) has shown that this is true at least for 3×3 games, if one uses the same tie-breaking rule as Krishna (1992). While we cannot prove the conjecture, we can extend Hahn's result to $3\times m$ and 4×4 NDQSMGs. Basically, these games have the CFPP because they have the FIP, and hence an ordinal potential.

Theorem 20 *Every* $3\times m$ *NDQSMG has the CFPP.*

Proof. See Appendix A.

Theorem 21 *Every* 4×4 *NDQSMG has the CFPP.*

Proof. See Appendix B.

Unfortunately, this method of proof does not extend to NDQSMGs with 'more' strategies. An example of a 4×5 NDQSMG with the improvement cycle $(1,4)\to(1,3)\to(2,3)\to(2,1)\to(2,5)\to(3,5)\to(3,2)\to(4,2)\to(4,4)\to(1,4)$ is

$$(A,B)=\begin{bmatrix} 3,4 & 3,3 & 2,2 & 2,1 & 0,0 \\ 2,2 & 2,4 & 3,1 & 3,0 & 1,3 \\ 1,2 & 0,4 & 0,1 & 0,0 & 2,3 \\ 0,0 & 1,1 & 1,2 & 1,3 & 3,4 \end{bmatrix}.$$

7 Discussion

In this paper, we have combined the Improvement Principle of Monderer and Sela (1997) with what we called the Second Improvement Principle to show that a CFP path essentially follows an improvement path. It is then an easy consequence that in NDOPGs, since these games have the FIP, every CFP path converges. The observation that improvement steps along a CFP path in games with DR have length 1 allowed us to prove the CFPP for NDQSMGs with DR. Finally, $3\times m$ and 4×4 NDQSMGs could be shown to have the FIP, and hence the CFPP. These four main theorems extend the respective results of Monderer and Shapley (1996), Krishna (1992), and Hahn (1999).

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Fig. A.1. The construction in the proof of Theorem 20.

A Proof of Theorem 20

Let (A, B) be a $3 \times m$ NDQSMG, and assume w.l.o.g. that there are no dominated strategies. Then $(1, 1)$ and $(3, m)$ are equilibria, and $(3, 1) \to (2, 1) \to (1, 1)$ and $(1, m) \to (2, m) \to (3, m)$. Also, $(1, k) \to (1, k-1)$ for all $k = 2, \ldots, m$, and $(3, k) \to (3, k+1)$ for all $k = 1, \ldots, m-1$. Assume there is an improvement cycle.

Suppose there is an improvement step from $(3, j'')$ to $(1, j'')$ in the improvement cycle. Then by quasi-supermodularity, $(3, j) \to (1, j)$ for $j = 1, \ldots, j''$. In the next step, the cycle must move 'left' or 'down' in Figure A.1. Since the cycle cannot reach the equilibrium $(1, 1)$, there must eventually be a step down from, say, $(1, j')$. This step cannot be to line 3, since $(3, j') \to (1, j')$. Hence it is $(1, j') \to (2, j')$, as in Case a) of Figure A.1. Then by quasi-supermodularity, $(1, j'') \to (2, j'')$. This implies $(3, j'') \to (2, j'')$. Hence $(3, j) \to (2, j)$ for $j = 1, \ldots, j''$. But this means that if we continue the improvement path *backwards* from $(3, j'')$, we move to the left and can never leave line 3, ending up in $(3, 1)$. This is a contradiction, since $(3, 1)$ cannot be part of an improvement cycle. If, on the other hand, there is a step $(2, j'') \to (1, j'')$ in the cycle, then we also have $(2, j') \to (1, j')$, and the improvement step leaving line 1 must be $(1, j') \to (3, j')$, see Case b) of Figure A.1. This implies $(2, j') \to (3, j')$. Hence $(1, j) \to (3, j)$ and $(2, j) \to (3, j)$ for $j = j', \ldots, m$. But then the improvement path moves to the right from $(3, j')$ and can never leave line 3, ending up in the equilibrium $(3, m)$. This is again a contradiction, since $(3, m)$ cannot be part of an improvement cycle. This means the game has the FIP, and hence an ordinal potential. By Theorem 15 then, it has the CFPP. □

B Proof of Theorem 21

Let (A, B) be a 4×4 NDQSMG, and assume w.l.o.g. that there are no dominated strategies. Then $(1,1)$ and $(4,4)$ are equilibria, and $(4,1) \to (3,1) \to (2,1) \to (1,1)$ and $(1,4) \to (2,4) \to (3,4) \to (4,4)$. Similarly, $(1,4) \to (1,3) \to (1,2) \to (1,1)$ and $(4,4) \to (4,3) \to (4,2) \to (4,1)$. Assume now that there is an improvement cycle C. By Theorem 20, a 3×4 NDQSMG has no improvement cycles, hence C must involve all four lines and all four columns of the bimatrix (A, B). An improvement cycle cannot involve the equilibria $(1,1)$ and $(4,4)$. Moreover, it cannot involve the pairs $(1,4)$ and $(4,1)$, since no improvement step leads to one of these pairs. Hence C contains $(1,2)$ or $(1,3)$. Assume it contains $(1,3)$, and the next step leads to the pair $(k,3)$. Then C involves an improvement path of the form $(i,3) \to (1,3) \to (k,3)$, where $i, k \geq 2$. But then replacing this part of C by $(i,3) \to (k,3)$ creates a shorter improvement cycle, which does not involve the first line of the bimatrix—which is a contradiction. Hence, if C contains $(1,3)$, then it contains the improvement step $(1,3) \to (1,2)$. An analogous argument shows that if C contains $(1,2)$, then it contains $(1,3) \to (1,2)$. This means that in any case $(1,3) \to (1,2)$ is a part of C. Applying the same reasoning to the last line of the bimatrix, we can conclude that C also contains $(4,2) \to (4,3)$.

Next we introduce some notation for the possible orderings of payoffs in the second and third column of A:

Let $\mathbf{v} = (a_{22} - a_{12}, a_{32} - a_{22}, a_{42} - a_{32}, a_{32} - a_{12}, a_{42} - a_{22}, a_{42} - a_{12})$ and $\mathbf{w} = (a_{23} - a_{13}, a_{33} - a_{23}, a_{43} - a_{33}, a_{33} - a_{13}, a_{43} - a_{23}, a_{43} - a_{13})$.

From the last paragraph we know that C involves improvement paths of the form $(i,3) \to (1,3) \to (1,2) \to (k,2)$ and $(i',2) \to (4,2) \to (4,3) \to (k',3)$ for some $i, k \geq 2$ and $i', k' \leq 3$. This yields the rules R1 and R2 below. From the transitivity of the relation "greater than" we can deduce rules R3–R6. The last rule, R7, follows from quasi-supermodularity.

R1: $v_1 > 0$ or $v_4 > 0$ or $v_6 > 0$. $w_1 < 0$ or $w_4 < 0$ or $w_6 < 0$.

R2: $v_3 > 0$ or $v_5 > 0$ or $v_6 > 0$. $w_3 < 0$ or $w_5 < 0$ or $w_6 < 0$.

R3: $v_1 > 0, v_2 > 0 \Longrightarrow v_4 > 0$, and $v_1 < 0, v_2 < 0 \Longrightarrow v_4 < 0$.
$w_1 > 0, w_2 > 0 \Longrightarrow w_4 > 0$, and $w_1 < 0, w_2 < 0 \Longrightarrow w_4 < 0$.

R4: $v_2 > 0, v_3 > 0 \Longrightarrow v_5 > 0$, and $v_2 < 0, v_3 < 0 \Longrightarrow v_5 < 0$.
$w_2 > 0, w_3 > 0 \Longrightarrow w_5 > 0$, and $w_2 < 0, w_3 < 0 \Longrightarrow w_5 < 0$.

R5: $v_3 > 0, v_4 > 0 \Longrightarrow v_6 > 0$, and $v_3 < 0, v_4 < 0 \Longrightarrow v_6 < 0$.
$w_3 > 0, w_4 > 0 \Longrightarrow w_6 > 0$, and $w_3 < 0, w_4 < 0 \Longrightarrow w_6 < 0$.

R6: $v_1 > 0, v_5 > 0 \Longrightarrow v_6 > 0$, and $v_1 < 0, v_5 < 0 \Longrightarrow v_6 < 0$.
$w_1 > 0, w_5 > 0 \Longrightarrow w_6 > 0$, and $w_1 < 0, w_5 < 0 \Longrightarrow w_6 < 0$.

R7: $v_k > 0 \Longrightarrow w_k > 0$ for $1 \leq k \leq 6$.

Now we examine all possible distinct cases.

- Case A: $v_6 < 0$.
  - · Case A1: $v_1 < 0$. R1 $\Longrightarrow v_4 > 0$. R5 $\Longrightarrow v_3 < 0$. R2 $\Longrightarrow v_5 > 0$. R4 $\Longrightarrow v_2 > 0$. R7 $\Longrightarrow w_2, w_4, w_5 > 0$.
    - Case A11: $w_1 > 0$. R6 $\Longrightarrow w_6 > 0$, contradicting R1.
    - Case A12: $w_1 < 0$.
      - Case A121: $w_3 > 0$. R5 $\Longrightarrow w_6 > 0$, contradicting R2.
      - Case A122: $w_3 < 0$.
  - · Case A2: $v_1 > 0$. R6 $\Longrightarrow v_5 < 0$. R2 $\Longrightarrow v_3 > 0$. R5 $\Longrightarrow v_4 < 0$. R3 $\Longrightarrow v_2 < 0$. R7 $\Longrightarrow w_1, w_3 > 0$.
    - Case A21: $w_4 > 0$. R5 $\Longrightarrow w_6 > 0$, contradicting R1.
    - Case A22: $w_4 < 0$.
      - Case A121: $w_5 > 0$. R6 $\Longrightarrow w_6 > 0$, contradicting R2.
      - Case A122: $w_5 < 0$.
- Case B: $v_6 > 0$. R7 $\Longrightarrow w_6 > 0$.
  - · Case B1: $w_1 > 0$. R1 $\Longrightarrow w_4 < 0$. R5 $\Longrightarrow w_3 > 0$. R2 $\Longrightarrow w_5 < 0$. R4 $\Longrightarrow w_2 < 0$. R7 $\Longrightarrow v_2, v_4, v_5 < 0$. R5 $\Longrightarrow v_3 > 0$. R6 $\Longrightarrow v_1 > 0$.
  - · Case B2: $w_1 < 0$. R6 $\Longrightarrow w_5 > 0$. R2 $\Longrightarrow w_3 < 0$. R5 $\Longrightarrow w_4 > 0$. R3 $\Longrightarrow w_2 > 0$. R7 $\Longrightarrow v_1, v_3 < 0$. R5 $\Longrightarrow v_4 > 0$. R6 $\Longrightarrow v_5 > 0$. R4 $\Longrightarrow v_2 > 0$.

Let us ignore v_6 and w_6 for the moment and write $\mathbf{v}_A = \text{sgn}(v_1, \dots, v_5)$, and $\mathbf{w}_A = \text{sgn}(w_1, \dots, w_5)$, where $\text{sgn}(x) = +(-)$ if $x >(<) 0$. Then the analysis of the different cases above shows that there are only two possible combinations $(\mathbf{v}_A, \mathbf{w}_A)$: $\mathbf{v}_A = \mathbf{w}_A = (-+-++)$ and $\mathbf{v}_A = \mathbf{w}_A = (+-+--)$. Repeating this argument for player 2 instead of player 1, we get exactly the same possible payoff orderings in the second and third line of B. With the analogous notation for player 2, the two possibilities are $\mathbf{v}_B = \mathbf{w}_B = (-+-++)$ and $\mathbf{v}_B = \mathbf{w}_B = (+-+--)$. Hence, all in all there are four possible combinations of these payoff orderings in a 4×4 NDQSMG without dominated strategies. These orderings are indicated by arrows in Figure B.1. Note that in case (a), $(2, 2)$ and $(3, 3)$ cannot be contained in C, since there is no improvement step to $(2, 2)$, and $(3, 3)$ is a strict equilibrium. The same is true for the pairs $(2, 3)$ and $(3, 2)$ in case (b), for $(3, 2)$ and $(2, 3)$ in case (c), and for $(3, 3)$ and $(2, 2)$ in case (d).

Consider now cases (a) and (c). We know that C contains the improvement step $(4, 2) \to (4, 3)$. The improvement step following this one can only be $(4, 3) \to (1, 3)$, and by quasi-supermodularity (R7) this implies $(4, 2) \to (1, 2)$. But the improvement step preceding $(4, 2) \to (4, 3)$ can only be $(1, 2) \to (4, 2)$, which yields a contradiction. Next consider cases (b) and (d). C contains the improvement step $(1, 3) \to (1, 2)$. The improvement step following this one can only be $(1, 2) \to (4, 2)$, and R7 then implies $(1, 3) \to (4, 3)$. But the improvement step preceding $(1, 3) \to (1, 2)$ must be $(4, 3) \to (1, 3)$, which yields a contradiction. Hence an improvement cycle does not exist, and the game is an NDOPG. By Theorem 15, it has the CFPP. □

Fig. B.1. The construction in the proof of Theorem 21.