Amos-type bounds for modified Bessel function ratios.

Kurt Hornik and Bettina Grün

Original Citation:

Hornik, Kurt ORCID: https://orcid.org/0000-0003-4198-9911 and Grün, Bettina (2013)
Amos-type bounds for modified Bessel function ratios.
Journal of Mathematical Analysis and Applications, 408 (1).
pp. 91-101. ISSN 0022-247X
This version is available at: https://epub.wu.ac.at/5450/
Available in ePubWU: March 2017

License: Creative Commons Attribution 3.0 Austria (CC BY 3.0 AT)

ePubWU, the institutional repository of the WU Vienna University of Economics and Business, is provided by the University Library and the IT-Services. The aim is to enable open access to the scholarly output of the WU.

This document is the publisher-created published version.
Amos-type bounds for modified Bessel function ratios

Kurt Hornik a, Bettina Grün b,∗

a Institute for Statistics and Mathematics, WU Wirtschaftsuniversität Wien, Augasse 2–6, 1090 Vienna, Austria
b Department of Applied Statistics, Johannes Kepler University Linz, Altenbergerstraße 69, 4040 Linz, Austria

A R T I C L E I N F O

Article history:
Received 24 October 2012
Available online 4 June 2013
Submitted by Kathy Driver

Keywords: Boundedness
Modified Bessel function ratio
Modified Bessel functions of the first kind
Inequalities

A B S T R A C T

We systematically investigate lower and upper bounds for the modified Bessel function ratio \(R_v = I_{v+1}/I_v\), by functions of the form \(G_{\alpha,\beta}(t) = t/(\alpha + \sqrt{t^2 + \beta^2})\) in case \(R_v\) is positive for all \(t > 0\), or equivalently, where \(v \geq -1\) or \(v\) is a negative integer. For \(v \geq -1\), we give an explicit description of the set of lower bounds and show that it has a greatest element. We also characterize the set of upper bounds and its minimal elements. If \(v \geq -1/2\), the minimal elements are tangent to \(R_v\) in exactly one point \(0 \leq t \leq \infty\), and have \(R_v\) as their lower envelope. We also provide a new family of explicitly computable upper bounds. Finally, if \(v\) is a negative integer, we explicitly describe the sets of lower and upper bounds, and give their greatest and least elements, respectively.

© 2013 The Authors. Published by Elsevier Inc. Open access under CC BY license.

1. Introduction

Let \(I_v\) be the modified Bessel function of order \(v\), and \(R_v\) the (modified) Bessel function ratio \(R_v(t) = I_{v+1}(t)/I_v(t)\). These ratios are of great importance in a variety of application areas, including statistics [e.g., 7] and numerical analysis [e.g., 1], either directly or through the fact that by the well-known recurrence relations for modified Bessel functions,

\[
\log(I_v)'(t) = \frac{I_v'(t)}{I_v(t)} = \frac{I_{v+1}(t) + (v/t)I_v(t)}{I_v(t)} = R_v(t) + \frac{v}{t}
\]

from which by integration and taking limits,

\[
\log(I_v)(t) = \int_0^t R_v(s) \, ds + v \log(t/2) - \log(f'(v + 1)).
\]

For functions \(f\) and \(g\) defined on the positive reals, write \(f \leq g\) iff \(f(t) \leq g(t)\) for all \(t > 0\), with \(f < g\) defined analogously. If neither \(f \leq g\) nor \(g \leq f\), we say that \(f\) and \(g\) are incomparable. Let \(\mathcal{G}\) be a family of functions on the positive reals and \(f \in \mathcal{G}\). We say that \(f\) is the least element (minimum) of \(\mathcal{G}\) iff \(f \leq g\) for all \(g \in \mathcal{G}\), and that \(f\) is a minimal element of \(\mathcal{G}\) iff there is no \(g \in \mathcal{G}\) for which \(f > g\), with the greatest element (maximum) and maximal elements of \(\mathcal{G}\) defined analogously. Let

\[
G_{\alpha,\beta}(t) = \frac{t}{\alpha + \sqrt{t^2 + \beta^2}},
\]

where in what follows we always (without loss of generality) take \(\beta \geq 0\). For \(v \geq 0\), Eqs. (9), (11) and (16) in Amos [1] show that

\[
\max(G_{v+1,v+1}, G_{v+1/2,v+3/2}) \leq R_v \leq \min(G_{v,v}, G_{v,v+2}, G_{v+1/2,v+1/2}).
\]

∗ Corresponding author.
E-mail addresses: Kurt.Hornik@wu.ac.at (K. Hornik), Bettina.Gruen@jku.at (B. Grün).

0022-247X © 2013 The Authors. Published by Elsevier Inc. Open access under CC BY license.
http://dx.doi.org/10.1016/j.jmaa.2013.05.070
Such “Amos-type” bounds were re-established and extended in several publications (see Section 3 for details). These bounds are very attractive because they allow both for explicit inversion and integration. Thus, Amos-type bounds yield bounds (and approximations) also for $R_v^{-1}$ and the antiderivative of $R_v$ (equivalently, $I_v$ and its logarithm).

Let
\[ \mathcal{L}_v = \{ (\alpha, \beta) : \mathcal{G}_{\alpha, \beta} \leq R_v \}, \quad \mathcal{U}_v = \{ (\alpha, \beta) : \mathcal{G}_{\alpha, \beta} \geq R_v \} \]
be the set of all $(\alpha, \beta)$ for which $\mathcal{G}_{\alpha, \beta}$ is a lower/upper Amos-type bound for $R_v$, and write
\[ \mathcal{g}_{\mathcal{L}_v} = \{ \mathcal{G}_{\alpha, \beta} : (\alpha, \beta) \in \mathcal{L}_v \}, \quad \mathcal{g}_{\mathcal{U}_v} = \{ \mathcal{G}_{\alpha, \beta} : (\alpha, \beta) \in \mathcal{U}_v \}, \]
for the corresponding families of lower/upper Amos-type bounds for $R_v$.

In this paper, we investigate the structure of $\mathcal{g}_{\mathcal{L}_v}$ and $\mathcal{g}_{\mathcal{U}_v}$ under the condition that $R_v > 0$, or equivalently, $\nu \geq -1$ or $\nu$ a negative integer.

2. Preliminaries

Let
\[ v_v(t) = \frac{U_v(t)}{U_{v+1}(t)} = \frac{t}{R_v(t)} \]
and
\[ h_{\alpha, \beta}(t) = \alpha + \sqrt{t^2 + \beta^2} \]
so that $\mathcal{G}_{\alpha, \beta}(t) = t/h_{\alpha, \beta}(t)$.

Using, e.g., Watson [10, Formula 3.7.2],
\[ R_v(t) = \frac{1}{2} \sum_{n=0}^{\infty} t^{2n} \left( \frac{4^n n! \Gamma(n + \nu + 2)}{\Gamma(n + 1)} \right). \]

If $\nu \geq -1$, all coefficients in the numerator and denominator series are non-negative and eventually positive, and hence $R_v > 0$. If $\nu$ is a negative integer, the same is true; otherwise, $\lim_{t \to 0} v_v(t) = 2\Gamma(\nu + 2)/\Gamma(\nu + 1) = 2(\nu + 1)$ which is negative if $\nu < -1$, and hence $R_v(t) < 0$ for all sufficiently small positive $t$.

Using the asymptotic expansion of $I_v$ for large argument [10, e.g., Formula 7.23.2], one can show that for arbitrary $\nu$,
\[ R_v(t) = 1 - \frac{\nu + 1/2}{t} + \frac{\nu^2 - 1/4}{2t^2} + O(1/t^3), \quad t \to \infty, \tag{1} \]
see also Schou [7, Eq. (6), assuming $\nu \geq 0$].

As $h_{\alpha, \beta}$ is increasing with $h_{\alpha, \beta}(0) = \alpha + \beta$, we have $\mathcal{G}_{\alpha, \beta} > 0$ if $\alpha + \beta > 0$. Hence, when $\nu \geq -1$ or $\nu$ is a negative integer and $\alpha + \beta \geq 0$, $\mathcal{G}_{\alpha, \beta}$ is a (strict) upper or lower bound for $R_v$ if and only if $h_{\alpha, \beta}$ is a (strict) lower or upper bound for $v_v$, respectively.

Lemma 1. For $\nu \geq -1$,
\[ v_v(t) = 2(\nu + 1) + \frac{t^2}{2(\nu + 2)} + O(t^4), \quad t \to 0. \tag{2} \]

Proof. More generally, if $\nu$ is not a negative integer,
\[ v_v(t) = \frac{(t/2)^\nu \left( \frac{1}{\Gamma(\nu+1)} + \frac{t^2/4}{\Gamma(\nu+2)} + O(t^4) \right)}{(t/2)^{\nu+1} \left( \frac{1}{\Gamma(\nu+2)} + \frac{t^2/4}{\Gamma(\nu+3)} + O(t^4) \right)} = \frac{2(\nu + 1) + t^2/4 + O(t^4)}{1 + \frac{t^2}{4(\nu+2)} + O(t^4)} = 2(\nu + 1) + \frac{t^2}{2(\nu + 2)} + O(t^4), \quad t \to 0. \]

If $\nu = -k$ is a negative integer, $1/\Gamma(\nu + n + 1)$ vanishes for $n$ from 0 to $k - 1$, and hence
\[ v_{-k}(t) = 2 \cdot \frac{\sum_{n=0}^{\infty} t^{2n} \left( \frac{4^n n! \Gamma(n - k + 1)}{\Gamma(n + 1)} \right)}{\sum_{n=k-1}^{\infty} t^{2n} \left( \frac{4^n n! \Gamma(n - k + 2)}{\Gamma(n + 1)} \right)} = \frac{2 \frac{t^{2k}}{(4^k k!)} \frac{1 + \frac{t^2}{4(k+1)} + O(t^4)}{1 + \frac{t^2}{4k} + O(t^4)}}{t^{2(k-1)}(k-1)!} = \frac{t^2}{2k} + O(t^4), \quad t \to 0. \]

As for $\nu = -k = -1$ we have $2(\nu + 1) = 0$ and $\nu + 2 = 1 = k$, we can combine the two expansions to obtain the lemma. \(\square\)
Lemma 2. If $\beta > 0$,
\[
h_{\alpha, \beta}(t) = (\alpha + \beta) + \frac{t^2}{2\beta} + O(t^4), \quad t \to 0. \tag{3}
\]
For arbitrary $\alpha$ and $\beta \geq 0$,
\[
G_{\alpha, \beta}(t) = 1 - \frac{\alpha}{t} + \frac{2\alpha^2 - \beta^2}{2t^2} + O(t^{-3}), \quad t \to \infty. \tag{4}
\]

Proof. If $\beta > 0$, then
\[
\sqrt{t^2 + \beta^2} = \beta \sqrt{1 + (t/\beta)^2} = \beta \left(1 + \frac{t^2}{2\beta^2} + O(t^4)\right) = \beta + \frac{t^2}{2\beta} + O(t^4)
\]
for $t \to 0$, whence Eq. (3) by adding $\alpha$.
As $t \to \infty$, $\sqrt{1 + \beta^2/t^2} = 1 + \beta^2/(2t^2) + O(t^{-4})$ and thus
\[
G_{\alpha, \beta}(t) = \frac{1}{\alpha/t + \sqrt{1 + \beta^2/t^2}} = \frac{1}{1 + \alpha/t + \beta^2/(2t^2) + O(t^{-4})}
\]
\[
= 1 - \frac{\alpha}{t} + \frac{2\alpha^2 - \beta^2}{2t^2} + O(t^{-3}), \quad t \to \infty. \quad \Box
\]

Theorem 1. For arbitrary $\nu$, $G_{\alpha, \beta} \leq R_\nu$ or $G_{\alpha, \beta} \geq R_\nu$ are only possible when $\alpha \geq \nu + 1/2$ or $\alpha \leq \nu + 1/2$, respectively. If $\nu \geq -1$, then $G_{\alpha, \beta} \leq R_\nu$ or $G_{\alpha, \beta} \geq R_\nu$ are only possible when $\alpha + \beta \geq 2(\nu + 1)$ or $0 \leq \alpha + \beta \leq 2(\nu + 1)$, respectively.

Proof. The first assertion is immediate by comparing the expansions of $R_\nu$ and $G_{\alpha, \beta}$ for $t \to \infty$. If $\alpha + \beta < 0$, $h_{\alpha, \beta}$ has a unique zero $t > 0$, and $G_{\alpha, \beta}$ changes from $-\infty$ to $\infty$ at $t$. If $\nu \geq -1$, $R_\nu > 0$, so upper and lower $G_{\alpha, \beta}$ bounds necessarily must have $\alpha + \beta \geq 0$. The second assertion now follows by comparing the values of $\nu$ and $h_{\alpha, \beta}$ at $t = 0$. \quad \Box

Lemma 3. Let $\beta_1 < \beta_2$ and $\min(\alpha_1 + \beta_1, \alpha_2 + \beta_2) \geq 0$. Then $G_{\alpha_1, \beta_1} < G_{\alpha_2, \beta_2}$ iff $\alpha_1 + \beta_1 \geq \alpha_2 + \beta_2$, and $G_{\alpha_1, \beta_1} > G_{\alpha_2, \beta_2}$ iff $\alpha_1 \leq \alpha_2$. Otherwise, if $\alpha_1 > \alpha_2$ and $\alpha_1 + \beta_1 < \alpha_2 + \beta_2$ and
\[
t = \sqrt{((\beta_2 - \beta_1)^2 - (\alpha_1 - \alpha_2)^2)((\beta_2 + \beta_1)^2 - (\alpha_1 - \alpha_2)^2)}
\]
\[
= \frac{2(\alpha_1 - \alpha_2)}{((\beta_2 - \beta_1)^2 - (\alpha_1 - \alpha_2)^2)}
\]
\[
G_{\alpha_1, \beta_1}(s) > G_{\alpha_2, \beta_2}(s) \text{ for } 0 < s < t \text{ and } G_{\alpha_1, \beta_1}(s) < G_{\alpha_2, \beta_2}(s) \text{ for } s > t.
\]

Proof. Consider $\Delta = h_{\alpha_1, \beta_1} - h_{\alpha_2, \beta_2}$. Then $\Delta(0) = (\alpha_1 + \beta_1) - (\alpha_2 + \beta_2)$ and as
\[
\sqrt{t^2 + \beta_1^2} - \sqrt{t^2 + \beta_2^2} = \frac{(t^2 + \beta_1^2) - (t^2 + \beta_2^2)}{\sqrt{t^2 + \beta_1^2} + \sqrt{t^2 + \beta_2^2}} = \frac{\beta_1^2 - \beta_2^2}{\sqrt{t^2 + \beta_1^2} + \sqrt{t^2 + \beta_2^2}} \to 0
\]
as $t \to \infty$, $\Delta(t) \to \Delta(\infty) = \alpha_1 - \alpha_2$ as $t \to \infty$. As
\[
\Delta'(t) = \frac{t}{\sqrt{t^2 + \beta_1^2}} - \frac{t}{\sqrt{t^2 + \beta_2^2}},
\]
if $\beta_1 < \beta_2$ we have $\Delta' > 0$ and hence $\Delta > 0$ iff $\Delta(0) \geq 0$, and $\Delta < 0$ iff $\Delta(\infty) \leq 0$. As $\min(\alpha_1 + \beta_1, \alpha_2 + \beta_2) \geq 0$, $G_{\alpha_1, \beta_1} < G_{\alpha_2, \beta_2}$ (or $\geq$) iff $\Delta > 0$ (or $\leq$). Otherwise, i.e., iff $\alpha_1 > \alpha_2$ and $\alpha_1 + \beta_1 < \alpha_2 + \beta_2$. $\Delta$ has a unique zero $t^*$ in $(0, \infty)$, which can be determined as follows. Let $u = \sqrt{t^2 + \beta_1^2} > \beta_1$ so that $t = \sqrt{u^2 - \beta_1^2}$ and $t^2 + \beta_2^2 = u^2 + (\beta_2^2 - \beta_1^2)$, and $\Delta(t) = 0$ iff
\[
\alpha_1 + u - \alpha_2 = \frac{u^2}{(\beta_2^2 - \beta_1^2)^2} - \frac{\alpha_1 - \alpha_2}{2}.
\]
Taking squares,
\[
u^2 + 2(\alpha_1 - \alpha_2)u + (\alpha_1 - \alpha_2)^2 = u^2 + (\beta_2^2 - \beta_1^2)
\]
from which
\[
u = \frac{\beta_2^2 - \beta_1^2}{2(\alpha_1 - \alpha_2)} - \frac{\alpha_1 - \alpha_2}{2}.
\]
Then
\[
    u - \beta_1 = \frac{(\beta_2^2 - \beta_1^2) - (\alpha_1 - \alpha_2)^2}{2(\alpha_1 - \alpha_2)} - \beta_1 = \frac{(\beta_2 - \beta_1 - \alpha_1 + \alpha_2)(\beta_2 + \beta_1 + \alpha_1 - \alpha_2)}{2(\alpha_1 - \alpha_2)}.
\]

The numerator equals \((\alpha_2 + \beta_2) - (\alpha_1 + \beta_1))(\alpha_1 - \alpha_2) + (\beta_1 + \beta_2)) > 0\) so that indeed \(u > \beta_1\). Similarly,
\[
    u + \beta_1 = \frac{\beta_2^2 - \beta_1^2 + 2\beta_1(\alpha_1 - \alpha_2) - (\alpha_1 - \alpha_2)^2}{2(\alpha_1 - \alpha_2)} = \frac{(\beta_2 + \beta_1 - \alpha_1 + \alpha_2)(\beta_2 - \beta_1 + \alpha_1 - \alpha_2)}{2(\alpha_1 - \alpha_2)}
\]
so that with \(t^2 = u^2 - \beta_1^2 = (u - \beta_1)(u + \beta_1)\) we indeed obtain
\[
    t = \sqrt{\frac{(\beta_2 - \beta_1)^2 - (\alpha_1 - \alpha_2)^2)((\beta_2 + \beta_1)^2 - (\alpha_1 - \alpha_2)^2)}{2(\alpha_1 - \alpha_2)}
\]
for the unique solution of \(\Delta(t) = 0\) (and equivalently \(G_{\alpha_1,\beta_1}(t) = G_{\alpha_2,\beta_2}(t)\) on \((0, \infty)\). Clearly, \(\Delta(s) < 0\) for \(0 < s < t\) and \(\Delta(s) > 0\) for \(s > t\), so that \(G_{\alpha_1,\beta_1}(s) > G_{\alpha_2,\beta_2}(s)\) for \(0 < s < t\) and \(G_{\alpha_1,\beta_1}(s) < G_{\alpha_2,\beta_2}(s)\) for \(s > t\), and the proof is complete. \(\square\)

**Lemma 4.** Suppose the quadratic polynomial \(Q(t) = t^2 + \gamma t + \delta\) has two real zeros \(t_1 \leq t_2\). Then \(Q(t) < 0\) iff \(t_1 < t < t_2\).

**Proof.** Trivial, as \(Q(t) = (t - t_1)(t - t_2)\). \(\square\)

3. Previous work

Amos [1] gives the bounds
\[
    G_{v+1/2,v+3/2} \leq R_v \leq G_{v+1/2,v+1/2}, \quad v \geq 0
\]
(Eq. (16)) and
\[
    G_{v+1,v+1} \leq R_v \leq G_{v,v+2}, \quad v \geq 0
\]
(Eqs. (9) and (11)). Using Lemma 3 with \(\beta_1 = v + 1 < v + 3/2 = \beta_2\) and \(\alpha_1 + \beta_1 = 2v + 2 = \alpha_2 + \beta_2\) we see that the first lower bound is uniformly better (larger) than the second one, whereas again with Lemma 3, neither of the upper bounds \(G_{v+1,v+1/2}\) and \(G_{v,v+2}\) is uniformly better (smaller) than the other: in fact, with \(\alpha_1 - \alpha_2 = 1/2, \beta_2 - \beta_1 = 3/2\) and \(\beta_2 + \beta_1 = 2v + 5/2\), we get
\[
    t = \sqrt{\frac{(9/4 - 1/4)(4v^2 + 10v + 25/4 - 1/4)}{2 \cdot (1/2)}} = 2\sqrt{(v+1)(2v+3)},
\]
so that \(G_{v+1/2}(s) < G_{v+1,v+1/2}(s)\) for \(0 < s < t\) and \(G_{v+1/2,v+1/2}(s) < G_{v+1/2}(s)\) for \(s > t\).

Näsell [5] gives rational bounds for \(R_v\), and notes (p. 8) that the Amos-type bounds \(G_{v+1/2,v+3/2} < R_v\) and \(R_v < G_{v+1,v+1/2}\) are valid for \(v > 1\) and \(v > -1/2\), respectively. But trivially \(R_{-1/2} = \tanh < 1 = G_{0,0}\), so that the upper bound is in fact valid for \(v \geq -1/2\).

Simpson and Spector [9, Theorem 2] show that
\[
    v_v(t^2) - (2v + 1)v_v(t) - (t^2 + v + 1/2) > 0, \quad t > 0, \quad v \geq 0.
\]
As the quadratic function \(Q(s) = s^2 - (2v + 1)s - (t^2 + v + 1/2)\) has zeros
\[
    v + 1/2 \pm \sqrt{(v + 1/2)^2 + (t^2 + v + 1/2)} = v + 1/2 \pm \sqrt{t^2 + (v + 1/2)(v + 3/2)},
\]
**Lemma 4** implies that \(v_v(t) > v + 1/2 + \sqrt{t^2 + (v + 1/2)(v + 3/2)}\) and hence
\[
    R_v < G_{v+1/2,v+1/2}(v+3/2), \quad v \geq 0.
\]
Using Lemma 3, we see that this bound is uniformly better than the Amos-type bound \(G_{v+1/2,v+1/2}\). To compare with \(G_{v,v+2}\), note that
\[
    ((\beta_2 - \beta_1)^2 - (\alpha_1 - \alpha_2)^2)((\beta_2 + \beta_1)^2 - (\alpha_1 - \alpha_2)^2) = (\beta_2^2 - \beta_1^2)^2 - 2(\beta_2^2 + \beta_1^2)(\alpha_1 - \alpha_2)^2 + (\alpha_1 - \alpha_2)^4.
\]
Thus, using Lemma 3 with \(\alpha_1 = v + 1/2, \beta_1 = \sqrt{(v + 1/2)(v + 3/2)}, \alpha_2 = v\) and \(\beta_2 = v + 2\), we get \(\alpha_1 - \alpha_2 = 1/2, \beta_2 - \beta_1 = 2v + 13/4, \beta_2^2 + \beta_1^2 = 2v^2 + 6v + 19/4\) and
\[
    t = \sqrt{(2v + 13/4)^2 - 2(2v^2 + 6v + 19/4)/4 + 1/16} = \sqrt{3v^2 + 10v + 33/4} = \sqrt{(3v + 11/2)(v + 3/2)},
\]
and therefore \(G_{v+1/2,v+1/2}(v+3/2)(s) < G_{v,v+2}(s)\) for \(s > t\), and \(G_{v,v+2}(s) < G_{v+1/2,v+1/2}(v+3/2)(s)\) for \(0 < s < t\).
Neuman [6, Proposition 5] shows that
\[ v^2_v(t) - (2v + 1)v_1(t) - (t^2 + v + 1/2) < v + 3/2, \quad t > 0, \quad v > -3/2. \]
As the quadratic function \( Q(s) = s^2 - (2v + 1)s - (t^2 + 2(v + 1)) \) has zeros
\[ v + 1/2 \pm \sqrt{(v + 1/2)^2 + t^2 + 2(v + 1)} = v + 1/2 \pm \sqrt{t^2 + (v + 3/2)^2}, \]
Lemma 4 implies that \( v_v(t) < v + 1/2 + \sqrt{t^2 + (v + 3/2)^2} \) for \( t > 0 \) and \( v > -3/2 \). If \( v \geq -1, v_v > 0 \) and hence \( R_v > G_{v+1/2,v+3/2} \).
Yuan and Kalbfleisch [11, Eq. (A.5)] show that
\[ G_{0+1,v+1} \leq R_v \leq G_v, \quad v > -1. \]
Baricz and Neuman [2, Theorems 2.1 and 2.2] show that if \( a > 1 \) and \( b = 1/(4 \log(a)) \), then
\[ v_v(t)^2 - (2v + 1)v_1(t) - t^2 < 2(v + 1), \quad 0 < t \leq 2b, \quad v \geq b - 2 \]
and that
\[ v_v(t)^2 - 2v v_v(t) - t^2 > 4(v + 1), \quad t > 0, \quad v > -2 \]
(the reference uses \( p - 1 \) for \( v \)). The former extends the earlier result of Neuman [6] when \( v \leq -3/2 \), in which case the bounds are not valid for all \( t > 0 \). As \( s \leftrightarrow Q(s) = s^2 - 2vs - (t^2 + 4(v + 1)) \) has zeros
\[ v \pm \sqrt{v^2 + t^2 + 4(v + 1)} = v \pm \sqrt{t^2 + (v + 2)^2}, \]
Lemma 4 yields that for \( v \geq -1 \), the latter is equivalent to \( R_v < G_v, v+2 \), extending the previously established \( v \) range for this bound.
Laforgia and Natalini [4, Theorem 1.1] show that
\[ \frac{-v + \sqrt{t^2 + v^2}}{t} < \frac{I_v(t)}{I_{v-1}(t)}, \quad t > 0, \quad v \geq 0 \]
(the condition that \( t > 0 \) is not stated explicitly in the theorem, but given in Eq. (1.8) of the reference used in the proof). As
\[ \frac{\sqrt{t^2 + v^2} - v}{t} = \frac{(t^2 + v^2) - v^2}{t \left( \sqrt{t^2 + v^2} + v \right)} = \frac{t}{v + \sqrt{t^2 + v^2}} = G_v, v(t), \]
the result is equivalent to
\[ R_v > G_{v+1,v+1}, \quad v > -1, \]
which is weaker than the \( R_v > G_{v+1/2,v+3/2} \) bound.
Segura [8, Theorem 3] shows that
\[ \frac{I_{v+1/2}(t)}{I_{v-1/2}(t)} < \frac{t}{v + \sqrt{t^2 + v^2}}, \quad t > 0, \quad v \geq 0 \]
or equivalently, \( R_v < G_{v+1/2,v+1/2} \) for \( v \geq -1/2 \). For \( r_v(t) = I_v(t)/(I_{v-1}(t)) = R_{v-1}(t)/t, \) Segura [8, Eqs. (22) and (61)] also shows that for \( t > 0 \) and \( v \geq 0 \),
\[ \frac{1}{(v - 1/2) + \sqrt{t^2 + (v + 1/2)^2}} < r_v(t) < \frac{1}{v + \sqrt{v^2 + t^2 v/(v + 1)}}. \]
Clearly, the lower bound is equivalent to \( R_v > G_{v+1/2,v+3/2} \) for \( v \geq -1 \), and the upper bound to
\[ R_v(t) < \frac{t}{v + 1 + \sqrt{(v + 1)^2 + t^2(v + 1)/(v + 2)}} \]
for \( t > 0 \) and \( v \geq -1 \), which is weaker than the upper bound \( R_v < G_v,v+2 \).
Kokologiannaki [3, Theorem 2.1] shows that for \( f_v(t) = I_{v+1}(t)/(I_v(t)) = R_v(t)/t, \)
\[ -\frac{v + 1}{t^2} + \frac{(v + 1)^2}{t^4} + \frac{1}{t^2} < f_v(t) < -\frac{v + 1}{t^2} + \frac{(v + 1)^2}{t^4} + \frac{1}{t^2} + \frac{1}{4(v + 1)^2(v + 2)} \]
for \( t > 0 \) and \( v > -1 \). As
\[ -\frac{v + 1}{t} + \frac{(v + 1)^2}{t^2} + 1 = \frac{\sqrt{t^2 + (v + 1)^2} - (v + 1)}{t}. \]
the lower bound again is equivalent to \( R_v > G_{v+1,v+1} \) for \( v > -1 \). Write \( U_R(t) \) for the above upper bound and \( \gamma = 1/(4(v + 1)^2(v + 2)) \). \( U_R(t) \) is the larger root of the quadratic polynomial

\[
s \mapsto Q(s; t) = s^2 + \frac{2(v + 1)}{t^2} s - \frac{1}{t^2} - \gamma,
\]

so by Lemma 4, for any function \( s(t) \) with \( Q(s; t) < 0 \) for all \( t > 0 \) we have \( s < U_R \). Consider \( s(t) = G_{v,v+2}(t)/t \), and write \( \beta = v + 2 \). Then \( Q(s(t); t) < 0 \) iff

\[
\frac{1}{v + \sqrt{t^2 + \beta^2}} + \frac{2(v + 1)}{t^2} \frac{1}{v + \sqrt{t^2 + \beta^2}} < \frac{1}{t^2} + \gamma,
\]

which in turn is equivalent to

\[
(1 + \gamma t^2) \left( v + \sqrt{t^2 + \beta^2} \right)^2 - 2(v + 1) \left( v + \sqrt{t^2 + \beta^2} \right) - t^2 > 0.
\]

Let \( \xi = \sqrt{t^2 + \beta^2} - \beta \) so that \( t \neq 0 \) iff \( \xi > 0 \), \( t^2 = (\xi + \beta)^2 - \beta^2 = \xi(\xi + 2\beta), v + \sqrt{t^2 + \beta^2} = 2(v + 1) + \xi \), and the inequality becomes

\[
0 < P(\xi) = \gamma \xi^4 + \gamma (4(v + 1) + 2\beta) \xi^3 + (1 + 8(v + 1)\beta \gamma + 4(v + 1)^2 \gamma - 1) \xi^2
\]

\[
+ (4(v + 1) + 8(v + 1)^2 \beta \gamma - 2(v + 1) - 2\beta) \xi + (4(v + 1)^2 - 4(v + 1)^2).
\]

The coefficient of the linear term is 0, so that

\[
P(\xi) = \gamma \xi^2 (\xi^2 + (4(v + 1) + 2\beta) \xi + (8(v + 1) \beta + 4(v + 1)^2))
\]

and for \( v > -1 \) we have \( P(\xi) > 0 \) for \( \xi > 0 \). Thus, \( G_{v,v+2}(t)/t < U_R(t) \) for all \( t > 0 \). We thus have the following.

**Theorem 2.** For all \( t > 0 \) and \( v > -1 \),

\[
\frac{G_{v,v+2}(t)}{t} < -\frac{v + 1}{t^2} + \sqrt{\frac{(v + 1)^2}{t^4} + \frac{1}{t^2} + \frac{1}{4(v + 1)^2(v + 2)}}.
\]

Hence, the upper bound in Kokologiannaki [3, Theorem 2.1] is strictly weaker than the bound \( f_v(t) = R_v(t)/t < G_{v,v+2}(t)/t \).

The various results can be summarized as follows: the “best” (in the sense of not being uniformly weaker than other) Amos-type bounds for \( R_v \) currently available are

\[
G_{v+1/2,v+1/2} < R_v, \quad v \geq -1,
\]

\[
R_v < G_{v,v+2}, \quad v \geq -1,
\]

\[
R_v < G_{v+1/2,\sqrt{(v+1/2)(v+1/2)}}, \quad v > 0,
\]

\[
R_v < G_{v+1/2,v+1/2}, \quad -1/2 \leq v \leq 0.
\]

4. Results

**Theorem 3.** For \( v \geq -1 \),

\[
\mathcal{L}_v = \{(\alpha, \beta) : \alpha \geq v + 1/2, \alpha + \beta \geq 2(v + 1), \beta \geq 0\}
\]

and \( G_{v+1/2,v+1/2} \) is the maximum of the family \( g_{\alpha,\beta} \) of lower Amos-type bounds for \( R_v \).

**Proof.** We already know that for \( v \geq -1, G_{v+1/2,v+1/2} < R_v \). By Theorem 1, \( G_{\alpha,\beta} \leq R_v \) is only possible if \( \alpha + \beta \geq 2(v + 1) = (v + 1/2) + (v + 3/2) \) and \( \alpha \geq v + 1/2 \). If \( \beta < v + 3/2 \), Lemma 3 implies that \( G_{\alpha,\beta} < G_{v+1/2,v+1/2} \). Otherwise, we trivially have \( G_{\alpha,\beta} \leq G_{v+3/2,v+3/2} \leq G_{v+1/2,v+1/2} \).

**Theorem 4.** For \( v \geq -1, \mathcal{U}_v \) is a closed convex set.

**Proof.** For fixed \( t > 0 \), \( (\alpha, \beta) \mapsto h_{\alpha,\beta}(t) \) is continuous, linear in \( \alpha \), and satisfies \( \partial h_{\alpha,\beta}(t)/\partial \beta = \beta(t^2 + \beta^2)^{-1/2} \geq 0 \) and hence

\[
\frac{\partial^2 h_{\alpha,\beta}(t)}{\partial \beta^2} = (t^2 + \beta^2)^{-1/2} - \beta^2(t^2 + \beta^2)^{-3/2} = t^2(t^2 + \beta^2)^{-3/2} \geq 0
\]

and is thus convex. By Theorem 1, \( G_{\alpha,\beta} \geq R_v \) is only possible when \( \alpha + \beta \geq 0 \), for which it is equivalent to \( h_{\alpha,\beta} \leq v_v \). Hence,

\[
\mathcal{U}_v = \{t \mapsto ((\alpha, \beta) : h_{\alpha,\beta}(t) \leq v(t))\}
\]

is the intersection of closed convex sets, and thus a closed convex set. \( \square \)
Let
\[ V_v(\alpha) = \{ \beta : (\alpha, \beta) \in U_v \} \]
\[ \beta^*_v(\alpha) = \sup V_v(\alpha) \]
\[ \alpha^*_v = \sup \{ \alpha : V_v(\alpha) \neq \emptyset \}. \]

As \( \lim_{\beta \to -\infty} G_{\alpha,\beta}(t) = 0 \) for \( t > 0 \), clearly \( \beta^*_v(\alpha) < \infty \) for \( v \geq -1 \).

**Theorem 5.** For \( v \geq -1 \),
\[ U_v = \{ (\alpha, \beta) : \alpha \leq \alpha^*_v, \max(0, -\alpha) \leq \beta \leq \beta^*_v(\alpha) \}, \]

with \( \beta^*_v \) continuous, decreasing and concave.

**Proof.** For \( v \geq -1 \), we have \( \beta \in V_v(\alpha) \) iff \( \alpha + \beta \geq 0 \) and \( h_{\alpha,\beta} \leq v \). Thus, as \( h_{\alpha,\beta} \) is continuous and increasing in \( \beta \), if \( V_v(\alpha) \) is non-empty, it is the closed interval \( [\max(0, -\alpha), \beta^*_v(\alpha)] \). By **Lemma 3**, \( G_{\alpha-n,\beta+n} > G_{\alpha,\beta} \) for all \( n > 0 \), so \( \beta^*_v \) must be decreasing as long as \( V_v(\alpha) \) is non-empty. If \( \alpha_n, \beta_n = \beta^*_v(\alpha_n) \) is decreasing and non-negative and thus must have a finite limit \( \beta_{\infty} \). Taking limits in \( \alpha_n + \beta_n \geq 0 \) and \( h_{\alpha_n,\beta_n} \leq v \) implies that \( \alpha^*_v + \beta_{\infty} \geq 0 \) and \( h_{\alpha^*_v,\beta_{\infty}} \leq v \). Thus, \( V_v(\alpha^*_v) \) is non-empty. As \( U_v = \bigcup_0^\infty V_v(\alpha) \), the first assertion follows. Finally, as \( U_v \) is closed and convex, \( \beta^*_v \) must be continuous and concave. \( \Box \)

**Theorem 6.** Let \( v \geq -1 \). For \( \alpha \leq v, \beta^*_v(\alpha) = 2(v + 1) - \alpha \). For \( \alpha < \beta^*_v(\alpha) = 2(v + 1) - \alpha \).

**Proof.** We know that \( (v, v + 2) \in U_v \). By **Theorem 1**, \( G_{\alpha,\beta} \geq R_v \) is only possible if \( \alpha + \beta \leq 2(v + 1) = v + (v + 2) \) so that \( \beta^*_v(\alpha) = 2(v + 1) - \alpha \). If \( \alpha + \beta = 2(v + 1) \) and \( \beta > 0 \),
\[ h_{\alpha,\beta}(t) = 2(v + 1) + \frac{t^2}{2\beta} + O(t^4), \quad t \to 0 \]
by Eq. (3) and comparison with Eq. (2) shows that \( h_{\alpha,\beta} \leq v \) is only possible if in fact \( \beta \geq v + 2 > 0 \), or equivalently, if \( \alpha \leq 2(v + 1) - (v + 2) = v \). For \( \alpha < v \), **Lemma 3** implies that \( G_{\alpha,2(v+1)-\alpha} > G_{v,v+2} \geq R_v \), so that indeed \( \beta^*_v(\alpha) = 2(v + 1) - \alpha \). \( \Box \)

Let
\[ Q_{\alpha,\beta}(s) = \beta^2 + (2(v + 1) - \alpha^2 - \beta^2)s + 2(v + 1/2 - \alpha)s^2. \]

**Lemma 5.** Let \( \Delta = v_v - h_{\alpha,\beta} \). Then
\[ t \Delta'(t) = \frac{Q_{\alpha,\beta}(\sqrt{t^2 + \beta^2})}{\sqrt{t^2 + \beta^2}} + (2(v + 1) - v_v(t) - h_{\alpha,\beta}(t)) \Delta(t). \]

**Proof.** As shown in Simpson and Spector [9], \( v \) satisfies the Riccati equation \( tv'_v(t) = t^2 + 2(v + 1)v_v(t) - v_v(t^2) \) and clearly, \( h_{\alpha,\beta}(t) = t/\sqrt{t^2 + \beta^2} \). Hence, as \( v^2 = h^2 + (v^2 - h^2) = h^2 + (v - h)(v + h) \),
\[ tv'_v(t) = t^2 + 2(v + 1)(\Delta(t) + h_{\alpha,\beta}(t)) - (h_{\alpha,\beta}(t^2 + \Delta(t)) + h_{\alpha,\beta}(t)) \]
\[ = t^2 + 2(v + 1)h_{\alpha,\beta}(t) - h_{\alpha,\beta}(t^2 + (2(v + 1) - v_v(t) - h_{\alpha,\beta}(t)) \Delta(t) \]
with
\[ t^2 + 2(v + 1)h_{\alpha,\beta}(t) - h_{\alpha,\beta}(t^2) = t^2 + 2(v + 1) \left( \alpha + \sqrt{t^2 + \beta^2} - \left( \alpha^2 + 2\alpha\sqrt{t^2 + \beta^2} + t^2 + \beta^2 \right) \right) \]
\[ = 2(v + 1)\alpha - \alpha^2 - \beta^2 + 2(v + 1 - \alpha)\sqrt{t^2 + \beta^2} \]
so that
\[ t^2 + 2(v + 1)h_{\alpha,\beta}(t) - h_{\alpha,\beta}(t^2) - \frac{t^2}{\sqrt{t^2 + \beta^2}} = \frac{(2(v + 1)\alpha - \alpha^2 - \beta^2)\sqrt{t^2 + \beta^2} + 2(v + 1 - \alpha)(t^2 + \beta^2) - t^2}{\sqrt{t^2 + \beta^2}} \]
\[ = \frac{Q_{\alpha,\beta}(\sqrt{t^2 + \beta^2})}{\sqrt{t^2 + \beta^2}}, \]
whence the lemma. \( \Box \)
Let 
\[ \alpha^*_v = \min(v + 1/2, 2v + 1) \]

(so that \( \alpha^*_v \) equals \( v + 1/2 \) for \( v \geq -1/2 \) and \( 2v + 1 \) otherwise), and for \( -1 \leq v \leq 1 \) let
\[ \beta^*_v(\alpha) = \sqrt{2v + 1 - 2\alpha} + \sqrt{2v + 1 + 2\alpha} - \alpha^2 = \sqrt{(v + 1/2 - \alpha)(\alpha + 1)(2v + 1 - \alpha)} \]

(where the second expressions shows that \( \beta^*_v \) is well-defined).

**Lemma 6.** Let \( v \geq -1 \). Then \( \beta^*_v \) is strictly concave with \( \beta^*_v(v) = v + 2 \). \( \beta^*_v(\alpha^*_v) \) equals \( \sqrt{(v + 1/2)(v + 3/2)} \) if \( v \geq -1/2 \) and \( \sqrt{-2(v + 1/2)} \) if \(-1 \leq v \leq -1/2 \), and \( \alpha \mapsto \alpha + \beta^*_v(\alpha) \) is non-negative and decreasing.

**Proof.** The assertions about the values of \( \beta^*_v \) at \( v \) and \( \alpha^*_v \) are straightforward. If \( v = -1 \), \( \alpha^*_v = v \) and there is nothing left to prove. Hence, take \( v > -1 \). The second derivative of \( \alpha \mapsto \sqrt{f(\alpha)} \) is given by
\[ \frac{d^2\sqrt{f(\alpha)}}{d\alpha^2} = \frac{f''(\alpha)f(\alpha) - f'(\alpha)^2/2}{2\sqrt{f(\alpha)^3}}. \]

For \( f_1(\alpha) = 2(v + 1/2 - \alpha) \) and \( f_2(\alpha) = 2v + 1 + 2\alpha - \alpha^2 \) we have \( f_1'(\alpha) = -2, f_1''(\alpha) = 0, f_2'(\alpha) = 2(\alpha - v) \) and \( f_2''(\alpha) = -2 \), giving numerators \(-2 \) and \(-2(2v + 1 + 2\alpha - \alpha^2) - 4(\alpha - v)^2/2 = -2(v + 1)^2 < 0 \). Hence \( \beta^*_v \) is the sum of two strictly concave functions, and thus strictly concave. Clearly,
\[ \frac{d\beta^*_v(\alpha)}{d\alpha} = \frac{-1}{\sqrt{2v + 1 - 2\alpha}} + \frac{\alpha - v}{\sqrt{2v + 1 + 2\alpha - \alpha^2}} \]

with value \(-1 \) at \( \alpha = v \). By strict concavity, the derivative of \( \beta^*_v \) is decreasing, and hence less than \(-1 \) for \( \alpha > v \), so that the derivative of \( \alpha \mapsto f(\alpha) = \alpha + \beta^*_v(\alpha) \) is negative for \( \alpha > v \) and \( f \) is decreasing. It remains to show that \( f(\alpha^*_v) \geq 0 \). If \( v \geq -1/2 \), this is immediate from \( \alpha^*_v = v + 1/2 \geq 0 \). Otherwise, \( \alpha^*_v = 2v + 1 < 0 \) and \( f(\alpha^*_v) = 2v + 1 + \sqrt{-2(v + 1)} \), which is non-negative as \( 0 \leq (2v + 1) \leq 1 \). \( \square \)

**Theorem 7.** Let \( v \geq -1 \). Then for \( v \leq \alpha \leq \alpha^*_v \), \( G_{v, \beta^*_v}(\alpha) \geq R_v \).

**Proof.** The proof will be based on the ideas of Simpson and Spector [9]. Suppose \( \Delta \) is sufficiently often continuously differentiable on \([0, \infty)\) with \( \Delta(0) > 0 \). Suppose that for all \( t > 0 \), \( \Delta(t) = 0 \) implies that there exists a suitable odd \( k \) such that \( \Delta(t^l) = 0 \) for \( l < k \) and \( \Delta(t^k) > 0 \). Then \( \Delta(t) \geq 0 \) for all \( t \geq 0 \), as otherwise for \( s = \inf[t > 0 : \Delta(t) = 0] \) we would have \( \Delta(s - \epsilon) = \Delta(s)\epsilon^k/k! < 0 \) for all sufficiently small \( \epsilon > 0 \) and a suitable \( s^* \in (s, \epsilon) \), which is impossible.

In our case, \( \Delta = v - h_{\alpha, \beta} \), where \( \beta = \beta^*_v(\alpha) \). If \( \alpha = v \), we have \( \beta = v + 2 \) and we already know for \( v \geq -1 \) that \( G_{v, \beta} = G_{v, v + 2} \geq R_v \). By **Lemma 6**, \( \alpha + \beta^*_v(\alpha) \) is decreasing and hence maximal for \( \alpha = v \) with value \( 2v + 1 \). Thus, for \( \alpha > v \) we have \( \alpha + \beta^*_v(\alpha) < 2(v + 1) \), or equivalently, \( \Delta(0) > 0 \).

Write \( s(t) = \sqrt{t^2 + \beta^2} \). If \( \alpha = v + 1/2 \), which is only possible if \( v \geq -1/2 \), we have \( \beta = \sqrt{(v + 1/2)(v + 3/2)} \) and \( G_{v, \beta} = \beta^2 \) for all \( s \). If \( v = -1/2 \), we already know that \( R_{-1/2} = \tanh \leq G_{0,0} \). Otherwise, \( G_{v, \beta}(s) = \beta^2 > 0 \). If \( \Delta(t) = 0 \) for some \( t > 0 \), **Lemma 5** implies that \( \Delta'(t) = \beta^2/(ts(t)) > 0 \), completing the proof for this case.

Hence, consider the case where \( v < \alpha < v + 1/2 \). Solving \( Q_{v, \beta}(s) = 0 \) has discriminant
\[ (2(v + 1)\alpha - \alpha^2 - \beta^2) - 8(v + 1/2 - \alpha)^2 \]

\[ = \left(2(v + 1)\alpha - \alpha^2 - \beta^2 + 2\beta\sqrt{2v + 1 - 2\alpha}\right)\left(2(v + 1)\alpha - \alpha^2 - \beta^2 - 2\beta\sqrt{2v + 1 - 2\alpha}\right), \]

with \( \beta = \beta^*_v(\alpha) \) the largest root of the first factor. Hence, the discriminant vanishes, and with
\[ \sigma = -\frac{2(v + 1)\alpha - \alpha^2 - \beta^2}{4(v + 1/2 - \alpha)} = \frac{2\sqrt{2v + 1 - 2\alpha}}{4(v + 1/2 - \alpha)} = \frac{\beta}{\sqrt{2v + 1 - 2\alpha}} > 0 \]

we have \( Q_{v, \beta}(s) = \gamma(s - \sigma)^2 \), where \( \gamma = 2v + 1 - 2\alpha > 0 \).

If \( \Delta(t) = 0 \) for some \( t > 0 \), **Lemma 5** implies that \( t\Delta'(t) = Q_{v, \beta}(s(t))/s(t) \). If \( s(t) \neq \sigma \), \( Q_{v, \beta}(s(t)) > 0 \), and the proof is complete. Otherwise, use **Lemma 5** to write \( t\Delta'(t) = \xi(t) + \eta(t)\Delta(t) \), where
\[ \xi(t) = \gamma(s(t) - \sigma)/s(t) = \gamma\left(s(t) - 2\sigma + \frac{\sigma^2}{s(t)}\right) \]

so that \( \xi(t) = \gamma(s'(t) - \sigma^2s'(t)/s(t)^2) \) and
\[ \xi''(t) = \gamma\left(s''(t) - \sigma^2\left(s'(t)^2 - 2s'(t)^2/s(t)^3\right)\right). \]
If $s(t) = \sigma, \xi'(t) = 0$ and $\xi''(t) = 2\gamma s'(t)^2/\sigma > 0$. Differentiation gives $\Delta'(t) + t \Delta''(t) = \xi'(t) + \eta'(t) \Delta(t) + \eta(t) \Delta'(t)$ and $2 \Delta''(t) + t \Delta'''(t) = \xi''(t) + \eta''(t) \Delta(t) + 2 \eta'(t) \Delta'(t) + \eta(t) \Delta''(t)$, so that if $s(t) = \sigma$, $\Delta(t) = \Delta'(t) = \Delta''(t) = 0$ and $\Delta'''(t) = \xi''(t)/t > 0$, and the proof is complete. □

**Theorem 8.** Let $v \geq -1$. Then the elements of $\{G_{\alpha,\beta^*_v(\alpha)} : v \leq \alpha \leq \alpha^*_v\}$ are mutually incomparable.

**Proof.** By Lemma 6, $\alpha \mapsto \alpha + \beta^*_v(\alpha)$ is decreasing, whence the result by using Lemma 3. □

**Theorem 9.** For $v \geq -1/2, \alpha^*_v = v + 1/2$ and
\[ \beta^*_v(v + 1/2) = \beta^*_v(v + 1/2) = \frac{\sqrt{(v + 1/2)(v + 3/2)}}{2} \]
For $-1 \leq v < -1/2, \alpha^*_v < v + 1/2$.

**Proof.** Let $\beta^* = \beta^*_v(v + 1/2) = \sqrt{(v + 1/2)(v + 3/2)}$. For arbitrary $\beta$,
\[ G_{v+1/2,\beta} = 1 - \frac{v + 1/2}{t^2} + \frac{2(v + 1/2)^2 - \beta^2}{2t^2} + O(t^{-3}), \quad t \to \infty \]
by Eq. (4) and comparison with Eq. (1) shows that $G_{v+1/2,\beta} \geq R_v$ is only possible if
\[ 2(v + 1/2)^2 - \beta^2 \geq (v + 1/2)(v - 1/2), \]
or equivalently, if $\beta^2 \leq 2(v + 1/2)^2 - (v + 1/2)(v - 1/2) = (v + 1/2)(v + 3/2)/2$. For $v < -1/2$, the upper bound is negative, so that $G_{v+1/2,\beta} \geq R_v$ is impossible for all $\beta \geq 0$ and hence $\alpha^*_v < v + 1/2$. For $v \geq 1/2$, the condition is equivalent to $\beta \leq \beta^*$. By Theorem 7, $(v + 1/2, \beta^*) \in U_v$ and by Theorem 1, $\alpha \leq v + 1/2$, so that $\alpha^*_v = v + 1/2$ and $\beta^*_v(v + 1/2) = \beta^*$. □

**Theorem 10.** Let $v \geq -1/2$ and $v < \alpha < v + 1/2$. Then there exists a unique positive $t^*_v(\alpha)$ at which $G_{\alpha,\beta^*_v(\alpha)}$ is tangent to $R_v$. The map $\alpha \mapsto t^*_v(\alpha)$ is continuous and increasing on $(v, v + 1/2)$, with $\lim_{\alpha \to v} t^*_v(\alpha) = 0$ and $\lim_{\alpha \to v + 1/2} t^*_v(\alpha) = \infty$.

**Proof.** Write $\beta^* = \beta^*_v(\alpha)$. By Theorem 6, we can find $\delta > 0$ such that $\beta^* \leq 2(v + 1) - \alpha - \delta$. Using Lemma 3 and the fact that $\sqrt{(v + 1/2)(v + 3/2)} \leq \beta^*_v(\alpha) \leq \beta^*$, we can find $0 < t_1 < t_2$ such that for all $\beta^* \leq \beta \leq \beta^* + \delta$, $G_{\alpha,\beta}(t) \geq G_{v,v+2}(t) > R_v(t) > 0$ for $0 < t < t_1$ and $G_{\alpha,\beta}(t) \geq G_{v+1/2,\sqrt{(v + 1/2)(v + 3/2)}}(t) > R_v(t)$ for $t \geq t_2$. If $G_{\alpha,\beta^*} > R_v$, we have for all $\eta > 0$ sufficiently small that $G_{\alpha,\beta^*+\eta}(t) \geq R_v(t)$ for $t_1 < t < t_2$. By the above, the same holds true for $0 < t < t_1$ and $t \geq 2$. Hence, $G_{\alpha,\beta^*+\eta} \geq R_v$ for all $\eta > 0$ sufficiently small, which contradicts the maximality of $\beta^*$. Thus, there must be at least one $t > 0$ such that $G_{\alpha,\beta^*}(t) = R_v(t)$, and clearly, the derivatives must agree at $t$ as otherwise $G_{\alpha,\beta^*}$ could not be an upper bound for $R_v$. Equivalently, $h_{\alpha,\beta}$ must be tangent to $v_v$ at $t$. By Lemma 5, this is the case iff $t$ solves $Q_{\alpha,\beta^*}(\sqrt{t^2 + \beta^*_v}) = 0$, from which we infer that $t = t^*_v(\alpha)$ is uniquely determined and continuous as a function of $\alpha$. The limits for $\alpha \to v$ from the right and $\alpha \to v + 1/2$ from the left are obvious. To show that $t^*\alpha$ is increasing, it suffices to show that it is injective. Hence, let $v < \alpha_1 < \alpha_2 < v + 1/2$ and suppose that $t^*_v(\alpha_1) = t^*_v(\alpha_2) = t^*$. Then with $\beta^*_v = \beta^*_v(\alpha_1)$, the $h_{\alpha,\beta^*_v}$ must have the same value and derivative at $t^*$, so that
\[ \frac{t^*}{\sqrt{t^*}} = \frac{t^*_v}{\sqrt{t^*_v}}, \]
and hence $h_{\alpha_1,\beta^*_v} = h_{\alpha_2,\beta^*_v}$, which is impossible as $\beta^*_v$ is decreasing by Theorem 5. □

**Theorem 11.** Let $v \geq -1/2$. Then $\{G_{\alpha,\beta^*_v(\alpha)} : v \leq \alpha \leq v + 1/2\}$ are the minimal elements of the family $g_{U_v}$ of upper Amos-type bounds for $R_v$, and
\[ R_v = \min\{G_{\alpha,\beta^*_v(\alpha)} : v \leq \alpha \leq v + 1/2\}. \]

**Proof.** Let $t > 0$. By Theorem 10, there exists a unique $v < \alpha < v + 1/2$ so that $t^*_v(\alpha) = t$ and hence $R_v(t) = G_{\alpha,\beta^*_v(\alpha)}(t)$, proving the second assertion. Let $v \leq \alpha_1 < \alpha_2 \leq v + 1/2$ and $\beta^*_1 = \beta^*_v(\alpha_1)$. If $\alpha_1 = v$, Theorem 6 shows that $2(v + 1) = \alpha_1 + \beta^*_1 > \alpha_2 + \beta^*_2$. If $\alpha_1 > v$ and $\alpha_1 + \beta^*_1 \leq \alpha_2 + \beta^*_2$, Lemma 3 implies that $R_v \leq G_{\alpha_1,\beta^*_1(\alpha_1)} < G_{\alpha_1,\beta^*_1(\alpha_1)}$, which is impossible as by Theorem 10, $G_{\alpha,\beta^*_1}$ must be the only tangent to $R_v$ at $t^*_v(\alpha_1)$. Thus we always have $\alpha_1 + \beta^*_1 > \alpha_2 + \beta^*_2$, and again by Lemma 3, there always exists $t = t(\alpha_1, \alpha_2)$ such that $G_{\alpha_1,\beta^*_1(\alpha_1)}(s) < G_{\alpha_2,\beta^*_2(\alpha_2)}(s)$ for $0 < s < t$ and $G_{\alpha_1,\beta^*_1(\alpha_1)}(s) > G_{\alpha_2,\beta^*_2(\alpha_2)}(s)$ for $s > t$. As $G_{\alpha,\beta^*_v(\alpha)} > G_{v,v+2} = G_{\alpha,\beta^*_v(\alpha)}$ for $\alpha < v$ and trivially $G_{\alpha,\beta} \geq G_{\alpha,\beta^*_v(\alpha)}$ provided that $(\alpha, \beta) \in U_v$, the first assertion follows, and the proof is complete. □
Finally, let us consider the cases where $\nu = -k$ is a negative integer. As readily seen from the series expansion, $I_{-k} = I_k$, and hence $R_{-k} = I_{-k+1}/I_{-k} = I_{k-1}/I_k = 1/R_{k-1}$.

**Theorem 12.** If $k$ is a positive integer,

$$\mathcal{U}_{-k} = \{(-\beta, \beta) : \beta \geq k\}$$

and $G_{-k,k}$ is the minimum of the family $\mathcal{g}_{U_k}$ of upper Amos-type bounds for $R_{-k}$.

**Proof.** As $R_{-k} > 0$ and has a pole at $t = 0$, the same must be true for upper bounds $G_{\alpha, \beta}$ of $R_{-k}$, implying that necessarily $\alpha + \beta = 0$. As

$$G_{-\beta, \beta}(t) = \frac{t}{\sqrt{t^2 + \beta^2} - \beta} = \frac{t (\sqrt{t^2 + \beta^2} + \beta)}{(t^2 + \beta^2) - \beta^2} = \frac{\sqrt{t^2 + \beta^2} + \beta}{t} = \frac{1}{G_{\beta, \beta}(t)},$$

we have $1/G_{\beta, \beta} = G_{-\beta, \beta} \geq R_{-k} = 1/R_{k-1}$ iff $R_{k-1} \geq G_{\beta, \beta}$, i.e., $(\beta, \beta) \in \mathcal{L}_{k-1}$. From the characterization of $\mathcal{L}_\nu$ for $\nu \geq -1$ (Theorem 3), this is possible iff $\beta \geq k - 1/2$ and $2\beta \geq 2k$, or equivalently, $\beta \geq k$. $\blacksquare$

**Theorem 13.** If $k$ is a positive integer,

$$\mathcal{L}_{-k} = \{(\alpha, \beta) : \alpha \geq -(k - 1/2), \alpha + \beta \geq 0, \beta \geq 0\}$$

and $G_{-(k-1)/2, k-1/2}$ is the maximum of the family $\mathcal{g}_{L_k}$ of lower Amos-type bounds for $R_{-k}$.

**Proof.** For lower bounds $G_{\alpha, \beta}$ of $R_{-k}$, we must have $\alpha + \beta \geq 0$ by the usual arguments, and Theorem 1 implies that necessarily $\alpha \geq -k + 1/2$. On the other hand, we also know that $R_{k-1} \leq G_{k-1/2,k-1/2}$, or equivalently, $G_{-(k-1)/2,k-1/2} \leq R_{-k}$, and the proof is complete. $\blacksquare$

Note that for $k = 1$, we already know by Theorem 3 that $G_{-1+1/2,-1+3/2} = G_{-1/2,1/2}$ is the greatest lower bound for $R_{-1}$, and Theorem 6 yields that $\beta_{-1,-1}^-(1) = 1$, so that $G_{-1,-1}$ is the least upper bound for $R_{-1}$ with $\alpha = -1$.

5. Summary and conclusions

In this paper, we systematically investigate lower and upper Amos-type bounds for $R_v = I_{v+1}/I_v$ on the positive reals when $R_v$ is positive, or equivalently, when $v \geq -1$ or $v$ is a negative integer.

For $v \geq -1$, the set $\mathcal{L}_v$ of all $(\alpha, \beta)$ giving lower bounds $G_{\alpha, \beta} \leq R_v$ has a simple explicit description, and $G_{v+1/2,v+3/2}$ is the maximum of the family $\mathcal{g}_{L_v}$ of lower Amos-type bounds for $R_v$ (Theorem 3).

For $v \geq -1$, the set $\mathcal{U}_v$ of all $(\alpha, \beta)$ giving upper bounds $G_{\alpha, \beta} \geq R_v$ is of the form $\{(\alpha, \beta) : \alpha \leq -\alpha_v, \max(0,-\alpha) \leq \beta \leq \beta_v(\alpha),\}$, where $\alpha_v^+ \leq v + 1/2$ and $\beta_v^-$ is continuous, decreasing and concave (Theorem 5), with $\beta_v^-(v) = v + 2$ and $\beta_v^+(v) = 2(v + 1)$ for $\alpha > v$ (Theorem 6). If $v \geq -1/2$, $\alpha_v^+ = v + 1/2$ and $\beta_v^+(v + 1/2) = \sqrt{v + 1/2}(v + 3/2)$ by Theorem 9, and the upper bounds in the family $\{G_{\alpha, \beta}(\alpha) : \alpha \leq \alpha_v^+ \leq v + 1/2\}$ are tangent to $R_v$ in exactly one point $t_v^*(\alpha)$ (Theorem 10, taking $t_v^*(v) = 0$ and $t_v^*(v + 1/2) = \infty$), and the minimal elements of the family $\mathcal{g}_{U_v}$ of upper Amos-type bounds for $R_v$, with $R_v$ as their lower envelope (Theorem 11).

Thus, for $v \geq -1$, the pointwise maximum over all lower Amos-type bounds equals $G_{v+1/2,v+3/2} = R_v$, and hence is always smaller than $R_v$. On the other hand, for $v \geq -1/2$, the pointwise minimum over all upper Amos-type bounds equals $R_v$.

For $v \geq -1$ and $v < \alpha < \alpha_v^+ = \min(v + 1/2, 2v + 1)$, Theorems 7 and 8 establish a family $\{G_{\alpha, \beta}(\alpha) : \alpha \leq \alpha_v^+ \}$ of explicitly computable, mutually incomparable upper bounds for $R_v$, with $\beta_v^+(v) = \beta_v^+(v + 2)$. For $v < \alpha < \alpha_v^+$, these bounds are new. For $v \geq -1/2$, $\alpha_v^+ = v + 1/2$ and $\beta_v^+(v + 1/2) = \beta_v^+(v + 1/2)$, and Theorem 7 extends the range of the bound $G_{v+1/2,\sqrt{v+1/2}(v+3/2)} = R_v$ given in Simpson and Spector [9] from $v \geq 0$ to $v \geq -1/2$, and for $-1/2 < v < 0$ dominates $G_{v+1/2,v+1/2} = R_v$ as the best previously available upper bound with $\alpha = v + 1/2$ (and hence first order exact as $t \to \infty$).

Finally, for the cases where $v = -k$ is a negative integer, Theorems 12 and 13 give explicit characterizations of $\mathcal{U}_{-k}$ and $\mathcal{L}_{-k}$, and establish $G_{-k,k}$ and $G_{-(k-1)/2,k-1/2}$ as the least upper and lower Amos-type bounds for $R_{-k}$, respectively.

For $-1 \leq v < -1/2$, the value of $\alpha_v^+$ is not known; the results in this paper imply that $\alpha_v^+ \leq \alpha_v^+ < v + 1/2$. It is also not known whether in this case $R_v$ can be obtained as the lower envelope of all upper Amos-type bounds. For $v = -1$, this is certainly not the case (as $G_{-1,1}$ is the uniformly smallest upper bound). Hence the range $-1 < v < -1/2$ deserves further investigation.

**Acknowledgment**

This research was funded by the Austrian Science Fund (FWF): V170-N18.
References


