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Amos-type bounds for modified Bessel function ratios

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**Abstract**

We systematically investigate lower and upper bounds for the modified Bessel function ratio \(R_\nu = I_{\nu+1}/I_\nu\) by functions of the form \(G_{\alpha,\beta}(t) = t/(\alpha + \sqrt{t^2 + \beta^2})\) in case \(R_\nu\) is positive for all \(t > 0\), or equivalently, where \(\nu \geq -1\) or \(\nu\) is a negative integer. For \(\nu \geq -1\), we give an explicit description of the set of lower bounds and show that it has a greatest element. We also characterize the set of upper bounds and its minimal elements. If \(\nu \geq -1/2\), the minimal elements are tangent to \(R_\nu\) in exactly one point \(0 \leq t \leq \infty\), and have \(R_\nu\) as their lower envelope. We also provide a new family of explicitly computable upper bounds. Finally, if \(\nu\) is a negative integer, we explicitly describe the sets of lower and upper bounds, and give their greatest and least elements, respectively.

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1. Introduction

Let \(I_\nu\) be the modified Bessel function of order \(\nu\), and \(R_\nu\) the (modified) Bessel function ratio \(R_\nu(t) = I_{\nu+1}(t)/I_\nu(t)\). These ratios are of great importance in a variety of application areas, including statistics [e.g., 7] and numerical analysis [e.g., 1], either directly or through the fact that by the well-known recurrence relations for modified Bessel functions,

\[
\log(I_\nu(t)) = \frac{I_{\nu+1}(t)}{I_\nu(t)} = \frac{I_{\nu+1}(t) + (\nu+1)I_\nu(t)}{I_\nu(t)} = R_\nu(t) + \frac{\nu}{t}
\]

from which by integration and taking limits,

\[
\log(I_\nu(t)) = \int_0^t R_\nu(s) \, ds + \nu \log(t/2) - \log(f'(\nu + 1)).
\]

For functions \(f\) and \(g\) defined on the positive reals, write \(f \leq g\) iff \(f(t) \leq g(t)\) for all \(t > 0\), with \(f < g\) defined analogously. If neither \(f \leq g\) nor \(g \leq f\), we say that \(f\) and \(g\) are incomparable. Let \(\bar{g}\) be a family of functions on the positive reals and \(f \in \bar{g}\). We say that \(f\) is the least element (minimum) of \(\bar{g}\) iff \(f \leq g\) for all \(g \in \bar{g}\), and that \(f\) is a minimal element of \(\bar{g}\) iff there is no \(g \in \bar{g}\) for which \(f > g\), with the greatest element (maximum) and maximal elements of \(\bar{g}\) defined analogously.

Let

\[
G_{\alpha,\beta}(t) = \frac{t}{\alpha + \sqrt{t^2 + \beta^2}}
\]

where in what follows we always (without loss of generality) take \(\beta \geq 0\). For \(\nu \geq 0\), Eqs. (9), (11) and (16) in Amos [1] show that

\[
\max(G_{\nu+1,\nu+1}, G_{\nu+1/2,\nu+3/2}) \leq R_\nu \leq \min(G_{\nu,\nu}, G_{\nu+2, \nu}, G_{\nu+1/2, \nu+1/2}).
\]

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Such “Amos-type” bounds were re-established and extended in several publications (see Section 3 for details). These bounds are very attractive because they allow both for explicit inversion and integration. Thus, Amos-type bounds yield bounds (and approximations) also for $R^{-1}_v$ and the antiderivative of $R_v$ (equivalently, $I_v$ and its logarithm).

Let

\[ \mathcal{L}_v = \{ (\alpha, \beta) : G_{\alpha, \beta} \leq R_v \}, \quad \mathcal{U}_v = \{ (\alpha, \beta) : G_{\alpha, \beta} \geq R_v \} \]

be the set of all $(\alpha, \beta)$ for which $G_{\alpha, \beta}$ is a lower/upper Amos-type bound for $R_v$, and write

\[ g_{\mathcal{L}_v} = \{ G_{\alpha, \beta} : (\alpha, \beta) \in \mathcal{L}_v \}, \quad g_{\mathcal{U}_v} = \{ G_{\alpha, \beta} : (\alpha, \beta) \in \mathcal{U}_v \}, \]

for the corresponding families of lower/upper Amos-type bounds for $R_v$.

In this paper, we investigate the structure of $g_{\mathcal{L}_v}$ and $g_{\mathcal{U}_v}$ under the condition that $R_v > 0$, or equivalently, $v \geq -1$ or $v$ a negative integer.

2. Preliminaries

Let

\[ v_v(t) = (t) / l_{v+1}(t) = t / R_v(t) \]

and

\[ h_{\alpha, \beta}(t) = \alpha + \sqrt{t^2 + \beta^2} \]

so that $G_{\alpha, \beta}(t) = t / h_{\alpha, \beta}(t)$.

Using, e.g., Watson [10, Formula 3.7.2],

\[ R_v(t) = \left[ \frac{\sum_{n=0}^{\infty} t^{2n} / (4^n n! \Gamma(n + v + 2))}{\sum_{n=0}^{2n} t^{2n} / (4^n n! \Gamma(n + v + 1))} \right]. \]

If $v \geq -1$, all coefficients in the numerator and denominator series are non-negative and eventually positive, and hence $R_v > 0$. If $v$ is a negative integer, the same is true; otherwise, $\lim_{t \to 0} v_v(t) = 2 \Gamma(v + 2) / \Gamma(v + 1) = 2(v + 1)$ which is negative if $v < -1$, and hence $R_v(t) < 0$ for all sufficiently small positive $t$.

Using the asymptotic expansion of $I_v$ for large argument [10, e.g., Formula 7.23.2], one can show that for arbitrary $v$,

\[ R_v(t) = 1 - \frac{\nu + 1/2}{t} + \frac{\nu^2 - 1/4}{2t^2} + O(1/t^3), \quad t \to \infty, \quad (1) \]

see also Schou [7, Eq. (6), assuming $v \geq 0$].

As $h_{\alpha, \beta}$ is increasing with $h_{\alpha, \beta}(0) = \alpha + \beta$, we have $G_{\alpha, \beta} > 0$ iff $\alpha + \beta > 0$. Hence, when $v \geq -1$ or $v$ is a negative integer and $\alpha + \beta \geq 0$, $G_{\alpha, \beta}$ is a (strict) upper or lower bound for $R_v$ if and only if $h_{\alpha, \beta}$ is a (strict) lower or upper bound for $v_v$, respectively.

**Lemma 1.** For $v \geq -1$,

\[ v_v(t) = 2(v + 1) + \frac{t^2}{2(v + 2)} + O(t^4), \quad t \to 0. \]

**Proof.** More generally, if $v$ is not a negative integer,

\[ v_v(t) = \left( \frac{(t/2)^{v+1} / \Gamma(v+1)}{(t/2)^{v+1} / \Gamma(v+2)} \right) \left( \frac{\Gamma(v+1)}{\Gamma(v+2)} + \frac{t^2/4}{\Gamma(v+3)} + O(t^4) \right) = 2 \left( \frac{v + 1}{1 + \frac{t^2}{4(v+2)}} + O(t^4) \right). \]

If $v = -k$ is a negative integer, $1 / \Gamma(v + n + 1)$ vanishes for $n$ from 0 to $k - 1$, and hence

\[ v_{-k}(t) = 2 \cdot \frac{\sum_{n=k}^{\infty} t^{2n} / (4^n n! \Gamma(n - k + 1))}{\sum_{n=k-1}^{\infty} t^{2n} / (4^n n! \Gamma(n - k + 2))} = \frac{2^{2k} / (4^k k!)}{1 + \frac{t^2}{4(k+1)} + O(t^4)} = \frac{t^2}{2k} + O(t^4), \quad t \to 0. \]

As for $v = -k = -1$ we have $2(v + 1) = 0$ and $v + 2 = 1 = k$, we can combine the two expansions to obtain the lemma. \[ \square \]
**Lemma 2.** If $\beta > 0$,

$$h_{\alpha, \beta}(t) = (\alpha + \beta) + \frac{t^2}{2\beta} + O(t^4), \quad t \to 0. \tag{3}$$

For arbitrary $\alpha$ and $\beta \geq 0$,

$$G_{\alpha, \beta}(t) = 1 - \frac{\alpha}{t} + \frac{2\alpha^2 - \beta^2}{2t^2} + O(t^{-3}), \quad t \to \infty. \tag{4}$$

**Proof.** If $\beta > 0$, then

$$\sqrt{t^2 + \beta^2} = \beta \sqrt{1 + (t/\beta)^2} = \beta \left(1 + \frac{t^2}{2\beta^2} + O(t^4)\right) = \beta + \frac{t^2}{2\beta} + O(t^4)$$

for $t \to 0$, whence Eq. (3) by adding $\alpha$.

As $t \to \infty$, $\sqrt{1 + \beta^2/t^2} = 1 + \beta^2/(2t^2) + O(t^{-4})$ and thus

$$G_{\alpha, \beta}(t) = \frac{1}{\alpha/t + \sqrt{1 + \beta^2/t^2}} = \frac{1}{1 + \alpha/t + \beta^2/(2t^2) + O(t^{-4})}$$

$$= 1 - \frac{\alpha}{t} + \frac{2\alpha^2 - \beta^2}{2t^2} + O(t^{-3}), \quad t \to \infty. \quad \Box$$

**Theorem 1.** For arbitrary $\nu$, $G_{\alpha, \beta} \leq R_\nu$ or $G_{\alpha, \beta} \geq R_\nu$ are only possible when $\alpha \geq \nu + 1/2$ or $\alpha \leq \nu + 1/2$, respectively. If $\nu \geq -1$, then $G_{\alpha, \beta} \leq R_\nu$ or $G_{\alpha, \beta} \geq R_\nu$ are only possible when $\alpha + \beta \geq 2(\nu + 1)$ or $0 \leq \alpha + \beta \leq 2(\nu + 1)$, respectively.

**Proof.** The first assertion is immediate by comparing the expansions of $R_\nu$ and $G_{\alpha, \beta}$ for $t \to \infty$. If $\alpha + \beta < 0$, $h_{\alpha, \beta}$ has a unique zero $t > 0$, and $G_{\alpha, \beta}$ changes from $-\infty$ to $\infty$ at $t$. If $\nu \geq -1$, $R_\nu > 0$, so upper and lower $G_{\alpha, \beta}$ bounds necessarily must have $\alpha + \beta \geq 0$. The second assertion now follows by comparing the values of $\nu$, and $h_{\alpha, \beta}$ at $t = 0$.\Box

**Lemma 3.** Let $\beta_1 < \beta_2$ and min$(\alpha_1 + \beta_1, \alpha_2 + \beta_2) \geq 0$. Then $G_{\alpha_1, \beta_1} < G_{\alpha_2, \beta_2}$ if $\alpha_1 + \beta_1 \geq \alpha_2 + \beta_2$, and $G_{\alpha_1, \beta_1} > G_{\alpha_2, \beta_2}$ if $\alpha_1 \leq \alpha_2$. Otherwise, if $\alpha_1 > \alpha_2$ and $\alpha_1 + \beta_1 < \alpha_2 + \beta_2$ and

$$t = \frac{\sqrt{(\beta_2 - \beta_1)^2 - (\alpha_1 - \alpha_2)^2}((\beta_2 + \beta_1)^2 - (\alpha_1 - \alpha_2)^2)}{2(\alpha_1 - \alpha_2)},$$

$G_{\alpha_1, \beta_1}(s) > G_{\alpha_2, \beta_2}(s)$ for $0 < s < t$ and $G_{\alpha_1, \beta_1}(s) < G_{\alpha_2, \beta_2}(s)$ for $s > t$.

**Proof.** Consider $\Delta = h_{\alpha_1, \beta_1} - h_{\alpha_2, \beta_2}$. Then $\Delta(0) = (\alpha_1 + \beta_1) - (\alpha_2 + \beta_2)$ and as

$$\sqrt{t^2 + \beta_1^2} - \sqrt{t^2 + \beta_2^2} = \frac{(t^2 + \beta_1^2) - (t^2 + \beta_2^2)}{\sqrt{t^2 + \beta_1^2} + \sqrt{t^2 + \beta_2^2}} = \frac{\beta_1^2 - \beta_2^2}{\sqrt{t^2 + \beta_1^2} + \sqrt{t^2 + \beta_2^2}} \to 0$$

as $t \to \infty$, $\Delta(t) \to \Delta(\infty) = \alpha_1 - \alpha_2$ as $t \to \infty$. As

$$\Delta'(t) = \frac{t}{\sqrt{t^2 + \beta_1^2}} - \frac{t}{\sqrt{t^2 + \beta_2^2}},$$

if $\beta_1 < \beta_2$ we have $\Delta' > 0$ and hence $\Delta > 0$ if $\Delta(0) \geq 0$, and $\Delta < 0$ if $\Delta(\infty) \leq 0$. As min$(\alpha_1 + \beta_1, \alpha_2 + \beta_2) \geq 0$, $G_{\alpha_1, \beta_1} < G_{\alpha_2, \beta_2}$ (or $\geq$) if $\Delta > 0$ (or $\leq$). Otherwise, i.e., if $\alpha_1 > \alpha_2$ and $\alpha_1 + \beta_1 < \alpha_2 + \beta_2$, $\Delta$ has a unique zero $t^*$ in $(0, \infty)$, which can be determined as follows. Let $u = \sqrt{t^2 + \beta_1^2} > \beta_1$ so that $t = \sqrt{u^2 - \beta_1^2}$ and $t^2 + \beta_2^2 = u^2 + (\beta_2^2 - \beta_1^2)$, and $\Delta(t) = 0$ iff

$$\alpha_1 + u - \alpha_2 = \sqrt{u^2 + (\beta_2^2 - \beta_1^2)}.$$
Then
\[ u - \beta_1 = \frac{(\beta_2^2 - \beta_1^2) - (\alpha_1 - \alpha_2)^2}{2(\alpha_1 - \alpha_2)} - \beta_1 = \frac{(\beta_2 - \beta_1 - \alpha_1 + \alpha_2)(\beta_2 + \beta_1 + \alpha_1 - \alpha_2)}{2(\alpha_1 - \alpha_2)}. \]
The numerator equals \((\alpha_2 + \beta_2) - (\alpha_1 + \beta_1))(\alpha_1 - \alpha_2) + (\beta_1 + \beta_2)) > 0\) so that indeed \(u > \beta_1\). Similarly,
\[ u + \beta_1 = \frac{\beta_2^2 - \beta_1^2 + 2\beta_1(\alpha_1 - \alpha_2) - (\alpha_1 - \alpha_2)^2}{2(\alpha_1 - \alpha_2)} = \frac{(\beta_2 + \beta_1 - \alpha_1 + \alpha_2)(\beta_2 - \beta_1 + \alpha_1 - \alpha_2)}{2(\alpha_1 - \alpha_2)} \]
so that with \(t^2 = u^2 - \beta_1^2 = (u - \beta_1)(u + \beta_1)\) we indeed obtain
\[ t = \frac{\sqrt{((\beta_2 - \beta_1)^2 - (\alpha_1 - \alpha_2)^2)((\beta_2 + \beta_1)^2 - (\alpha_1 - \alpha_2)^2)}}{2(\alpha_1 - \alpha_2)} \]
for the unique solution of \(\Delta(t) = 0\) (and equivalently \(G_{\alpha_1,\beta_1}(t) = G_{\alpha_2,\beta_2}(t)\) on \((0, \infty)\). Clearly, \(\Delta(s) < 0\) for \(0 < s < t\) and \(\Delta(s) > 0\) for \(s > t\), so that \(G_{\alpha_1,\beta_1}(s) > G_{\alpha_2,\beta_2}(s)\) for \(0 < s < t\) and \(G_{\alpha_1,\beta_1}(s) < G_{\alpha_2,\beta_2}(s)\) for \(s > t\), and the proof is complete. □

**Lemma 4.** Suppose the quadratic polynomial \(Q(t) = t^2 + \gamma t + \delta\) has two real zeros \(t_1 \leq t_2\). Then \(Q(t) < 0\) iff \(t_1 < t < t_2\).

**Proof.** Trivial, as \(Q(t) = (t - t_1)(t - t_2)\). □

3. Previous work

Amos [1] gives the bounds
\[
G_{\nu+1/2,\nu+3/2} \leq R_\nu \leq G_{\nu+1/2,\nu+1/2}, \quad \nu \geq 0
\]
(Eq. (16)) and
\[
G_{\nu+1,\nu+1} \leq R_\nu \leq G_{\nu,\nu}, \quad \nu \geq 0
\]
(Eqs. (9) and (11)). Using Lemma 3 with \(\beta_1 = \nu + 1 < \nu + 3/2 = \beta_2\) and \(\alpha_1 + \beta_1 = 2\nu + 2 = \alpha_2 + \beta_2\) we see that the first lower bound is uniformly better (larger) than the second one, whereas again with Lemma 3, neither of the upper bounds \(G_{\nu+1/2,\nu+1/2}\) and \(G_{\nu,\nu}\) is uniformly better (smaller) than the other: in fact, with \(\alpha_1 - \alpha_2 = 1/2, \beta_2 - \beta_1 = 3/2\) and \(\beta_2 + \beta_1 = 2\nu + 5/2\), we get
\[
t = \frac{\sqrt{(9/4 - 1/4)(4\nu^2 + 10\nu + 25/4 - 1/4)}}{2 \cdot (1/2)} = 2\sqrt{(\nu + 1)(2\nu + 3)},
\]
so that \(G_{\nu+2}(s) < G_{\nu+1/2,\nu+1/2}(s)\) for \(0 < s < t\) and \(G_{\nu+1/2,\nu+1/2}(s) < G_{\nu+1/2}(s)\) for \(s > t\).

Näsell [5] gives rational bounds for \(R_\nu\), and notes (p. 8) that the Amos-type bounds \(G_{\nu+1/2,\nu+3/2} < R_\nu\) and \(R_\nu < G_{\nu+1/2,\nu+1/2}\) are valid for \(\nu > -1\) and \(\nu > -1/2\), respectively. But trivially \(R_{-1/2} = \tanh < 1 = G_{0,0}\), so that the upper bound is in fact valid for \(\nu \geq -1/2\).

Simpson and Spector [9, Theorem 2] show that
\[
v_\nu(t^2 - (2\nu + 1)v_\nu(t) - (t^2 + 1/2) > 0, \quad t > 0, \quad \nu \geq 0.
\]
As the quadratic function \(Q(s) = s^2 - (2\nu + 1)s - (t^2 + 1/2)\) has zeros
\[
\nu + 1/2 \pm \sqrt{(\nu + 1/2)^2 + (t^2 + 1/2)} = \nu + 1/2 \pm \sqrt{t^2 + (\nu + 1/2)(\nu + 3/2),}
\]
**Lemma 4** implies that \(v_\nu(t) > \nu + 1/2 + \sqrt{t^2 + (\nu + 1/2)(\nu + 3/2)}\) and hence
\[
R_\nu < G_{\nu+1/2,\sqrt{(\nu+1/2)(\nu+3/2)}}(s) \quad \nu \geq 0.
\]
Using Lemma 3, we see that this bound is uniformly better than the Amos-type bound \(G_{\nu+1/2,\nu+1/2}\). To compare with \(G_{\nu,\nu+2}\), note that
\[
((\beta_2 - \beta_1)^2 - (\alpha_1 - \alpha_2)^2)((\beta_2 + \beta_1)^2 - (\alpha_1 - \alpha_2)^2) = (\beta_2^2 - \beta_1^2)^2 - 2(\beta_2^2 + \beta_1^2)(\alpha_1 - \alpha_2)^2 + (\alpha_1 - \alpha_2)^4.
\]
Thus, using Lemma 3 with \(\alpha_1 = \nu + 1/2, \beta_1 = \sqrt{(\nu + 1/2)(\nu + 3/2)}, \alpha_2 = \nu\) and \(\beta_2 = \nu + 2\), we get \(\alpha_1 - \alpha_2 = 1/2, \beta_2 - \beta_1 = 2\nu + 13/4, \beta_2^2 + \beta_1^2 = 2\nu^2 + 6\nu + 19/4\) and
\[
t = \sqrt{(2\nu + 13/4)^2 - 2(2\nu^2 + 6\nu + 19/4)/4 + 1/16} = \sqrt{3\nu^2 + 10\nu + 33/4} = \sqrt{(3\nu + 11/2)(\nu + 3/2)},
\]
and therefore \(G_{\nu+1/2,\sqrt{(\nu+1/2)(\nu+3/2)}}(s) < G_{\nu,\nu+2}(s)\) for \(s > t\) and \(G_{\nu,\nu+2}(s) < G_{\nu+1/2,\sqrt{(\nu+1/2)(\nu+3/2)}}(s)\) for \(0 < s < t\).
Neuman [6, Proposition 5] shows that
\[ v^2_v(t) - (2v + 1)v_0(t) - (t^2 + v + 1/2) < v + 3/2, \quad t > 0, \quad v > -3/2. \]

As the quadratic function \( Q(s) = s^2 - (2v + 1)s - (t^2 + 2(v + 1)) \) has zeros
\[ v + 1/2 \pm \sqrt{(v + 1/2)^2 + t^2 + 2(v + 1)} = v + 1/2 \pm \sqrt{t^2 + (v + 3/2)^2}, \]

**Lemma 4** implies that \( v_\nu(t) < v + 1/2 + \sqrt{t^2 + (v + 3/2)^2} \) for \( t > 0 \) and \( v > -3/2 \). If \( v \geq -1, v_\nu > 0 \) and hence \( R_\nu > G_{v+1/2,v+3/2} \).

Yuan and Kalbfleisch [11, Eq. (A.5)] show that
\[ G_{\nu+1,v+1} \leq R_\nu \leq G_{\nu,v}, \quad v > -1. \]

Baricz and Neuman [2, Theorems 2.1 and 2.2] show that if \( a > 1 \) and \( b = 1/(4 \log(a)) \), then
\[ v_\nu(t)^2 - (2v + 1)v_\nu(t) - t^2 < 2(v + 1), \quad 0 < t \leq 2b, \quad v \geq b - 2 \]

and that
\[ v_\nu(t)^2 - 2v_\nu(t) - t^2 > 4(v + 1), \quad t > 0, \quad v > -2 \]
(the reference uses \( p - 1 \) for \( v \)). The former extends the earlier result of Neuman [6] when \( v \leq -3/2 \), in which case the bounds are not valid for all \( t > 0 \). As \( s \mapsto Q(s) = s^2 - 2vs - (t^2 + 4(v + 1)) \) has zeros
\[ v \pm \sqrt{t^2 + v^2 + 4(v + 1)} = v \pm \sqrt{t^2 + (v + 2)^2}, \]

**Lemma 4** yields that for \( v \geq -1, \) the latter is equivalent to \( R_\nu < G_{v,v+2} \), extending the previously established \( v \) range for this bound.

Laforgia and Natalini [4, Theorem 1.1] show that
\[ -v + \sqrt{t^2 + v^2} \leq \frac{I_\nu(t)}{I_{\nu-1}(t)}, \quad t > 0, \quad v \geq 0 \]
(the condition that \( t > 0 \) is not stated explicitly in the theorem, but given in Eq. (1.8) of the reference used in the proof). As
\[ \frac{\sqrt{t^2 + v^2} - v}{t} = \frac{(t^2 + v^2) - v^2}{t(t^2 + v^2)} = \frac{t}{v + \sqrt{t^2 + v^2}} = G_{v,v}(t), \]

the result is equivalent to
\[ R_\nu > G_{\nu+1,v+1}, \quad v > -1, \]

which is weaker than the \( R_\nu > G_{\nu+1/2,v+3/2} \) bound.

Segura [8, Theorem 3] shows that
\[ \frac{I_{\nu+1/2}(t)}{I_{\nu-1/2}(t)} < \frac{t}{v + \sqrt{t^2 + v^2}}, \quad t > 0, \quad v \geq 0 \]
or equivalently, \( R_\nu < G_{\nu+1/2,v+1/2} \) for \( v \geq -1/2 \). For \( r_\nu(t) = I_\nu(t)/(tI_{\nu-1}(t)) = R_{\nu-1}(t)/t \), Segura [8, Eqs. (22) and (61)] also shows that for \( t > 0 \) and \( v \geq 0 \),
\[ \frac{1}{(v - 1/2) + \sqrt{t^2 + (v + 1/2)^2}} < r_\nu(t) < \frac{1}{v + \sqrt{v^2 + t^2v/(v + 1)}}. \]

Clearly, the lower bound is equivalent to \( R_\nu > G_{\nu+1/2,v+3/2} \) for \( v \geq -1 \), and the upper bound to
\[ R_\nu(t) < \frac{t}{v + 1 + \sqrt{(v + 1)^2 + t^2(v + 1)/(v + 2)}} \]
for \( t > 0 \) and \( v \geq -1 \), which is weaker than the upper bound \( R_\nu < G_{\nu,v+2} \).

Kokologiannaki [3, Theorem 2.1] shows that for \( f_\nu(t) = I_{\nu+1}(t)/(tI_\nu(t)) = R_\nu(t)/t \),
\[ -\frac{v + 1}{t^2} + \frac{1}{t^2} < f_\nu(t) < -\frac{v + 1}{t^2} + \frac{1}{t^2} + \frac{1}{4(v + 1)^2(v + 2)} \]
for \( t > 0 \) and \( v > -1 \). As
\[ -\frac{v + 1}{t} + \frac{1}{t^2} = \frac{\sqrt{t^2 + (v + 1)^2} - (v + 1)}{t}, \]

the lower bound again is equivalent to $R_v > G_{v+1,v+1}$ for $v > -1$. Write $U_K(t)$ for the above upper bound and $\gamma = 1/(4(v+1)^2(v+2))$. $U_K(t)$ is the larger root of the quadratic polynomial

$$s \mapsto Q(s; t) = s^2 + \frac{2(v+1)}{t^2} s - \frac{1}{t^2} - \gamma,$$

so by Lemma 4, for any function $s(t)$ with $Q(s(t), t) < 0$ for all $t > 0$ we have $s < U_K$. Consider $s(t) = G_{v,v+2}(t)/t$, and write $\beta = v + 2$. Then $Q(s(t), t) < 0$ iff

$$\frac{1}{(v + \sqrt{t^2 + \beta^2})^2} + \frac{2(v+1)}{t^2} \frac{1}{v + \sqrt{t^2 + \beta^2}} < \frac{1}{t^2} + \gamma,$$

which in turn is equivalent to

$$(1 + \gamma t^2) \left( v + \sqrt{t^2 + \beta^2} \right)^2 - 2(v+1) \left( v + \sqrt{t^2 + \beta^2} \right) - t^2 > 0.$$

Let $\xi = \sqrt{t^2 + \beta^2} - \beta$ so that $t \neq 0$ iff $\xi > 0$, $t^2 = (\xi + \beta)^2 - \beta^2 = \xi(\xi + 2\beta)$, $v + \sqrt{t^2 + \beta^2} = 2(v+1) + \xi$, and the inequality becomes

$$0 < P(\xi) = \gamma \xi^4 + \gamma(4(v+1) + 2\beta)\xi^3 + (1+8(v+1)\beta\gamma + 4(v+1)^2\gamma - 1)\xi^2$$

$$+ (4(v+1) + 8(v+1)^2\beta\gamma - 2(v+1) - 2\beta)\xi + (4(v+1)^2 - 4(v+1)^2).$$

The coefficient of the linear term is 0, so that

$$P(\xi) = \gamma \xi^4 + (4(v+1) + 2\beta)\xi^3 + (8(v+1)^2 + 4(v+1)^2))$$

and for $v > -1$ we have $P(\xi) > 0$ for $\xi > 0$. Thus, $G_{v,v+2}(t)/t < U_K(t)$ for all $t > 0$. We thus have the following.

**Theorem 2.** For all $t > 0$ and $v > -1$,

$$\frac{G_{v,v+2}(t)}{t} < -\frac{v+1}{t^2} + \frac{1}{\sqrt{\frac{(v+1)^2}{t^4} + \frac{1}{t^2} + \frac{1}{4(v+1)^2(v+2)}}}.$$

Hence, the upper bound in Kokologiannaki [3, Theorem 2.1] is strictly weaker than the bound $f_v(t) = R_v(t)/t < G_{v,v+2}(t)/t$.

The various results can be summarized as follows: the “best” (in the sense of not being uniformly weaker than other) Amos-type bounds for $R_v$ currently available are

- $G_{v+1/2,v+1/2} < R_v, \quad v \geq -1$,
- $R_v < G_{v,v+2}, \quad v \geq -1$,
- $R_v < G_{v+1/2,\sqrt{(v+1/2)(v+3/2)}}, \quad v > 0$,
- $R_v < G_{v+1/2,v+1/2}, \quad -1/2 \leq v \leq 0$.

4. Results

**Theorem 3.** For $v \geq -1$,

$$\mathcal{L}_v = \{ (\alpha, \beta) : \alpha \geq v + 1/2, \alpha + \beta \geq 2(v+1), \beta \geq 0 \}$$

and $G_{v+1/2,v+3/2}$ is the maximum of the family $g_{\alpha,\beta}$ of lower Amos-type bounds for $R_v$.

**Proof.** We already know that for $v \geq -1$, $G_{v+1/2,v+1/2} < R_v$. By Theorem 1, $G_{\alpha,\beta} \leq R_v$ is only possible if $\alpha + \beta \geq 2(v+1) = (v+1/2) + (v+3/2)$ and $\alpha \geq v+1/2$. If $\beta < v+3/2$, Lemma 3 implies that $G_{\alpha,\beta} < G_{v+1/2,v+3/2}$. Otherwise, we trivially have $G_{\alpha,\beta} \leq G_{v+1/2,v+3/2} \leq G_{v+1/2,v+1/2}$. □

**Theorem 4.** For $v \geq -1$, $U_v$ is a closed convex set.

**Proof.** For fixed $t > 0$, $(\alpha, \beta) \mapsto h_{\alpha,\beta}(t)$ is continuous, linear in $\alpha$, and satisfies $\partial h_{\alpha,\beta}(t)/\partial \beta = \beta(t^2 + \beta^2)^{-1/2} \geq 0$ and hence

$$\frac{\partial^2 h_{\alpha,\beta}(t)}{\partial \beta^2} = (t^2 + \beta^2)^{-1/2} - \beta^2(t^2 + \beta^2)^{-3/2} = t^2(t^2 + \beta^2)^{-3/2} \geq 0$$

and is thus convex. By Theorem 1, $G_{\alpha,\beta} \geq R_v$ is only possible when $\alpha + \beta \geq 0$, for which it is equivalent to $h_{\alpha,\beta} \leq v_v$. Hence, $U_v = \bigcap_{t>0} \{ (\alpha, \beta) : h_{\alpha,\beta}(t) \leq v(t) \}$ is the intersection of closed convex sets, and thus a closed convex set. □
Let
\[ \mathcal{V}_v(\alpha) = \{ \beta : (\alpha, \beta) \in \mathcal{U}_v \} \]
\[ \beta^*_v(\alpha) = \sup \mathcal{V}_v(\alpha) \]
\[ \alpha^*_v = \sup \{ \alpha : \mathcal{V}_v(\alpha) \neq \emptyset \}. \]

As \( \lim_{\beta \to \infty} G_{\alpha, \beta}(t) = 0 \) for \( t > 0 \), clearly \( \beta^*_v(\alpha) < \infty \) for \( \nu \geq -1 \).

**Theorem 5.** For \( \nu \geq -1 \),
\[ \mathcal{U}_v = \{ (\alpha, \beta) : \alpha \leq \alpha^*_v, \max(0, -\alpha) \leq \beta \leq \beta^*_v(\alpha) \}, \]
with \( \beta^*_v \) continuous, decreasing and concave.

**Proof.** For \( \nu \geq -1 \), we have \( \beta \in \mathcal{V}_v(\alpha) \) iff \( \alpha + \beta \geq 0 \) and \( h_{\alpha, \beta} \leq v_v \). Thus, as \( h_{\alpha, \beta} \) is continuous and increasing in \( \beta \), if \( \mathcal{V}_v(\alpha) \) is non-empty, it is the closed interval \([\max(0, -\alpha), \beta^*_v(\alpha)]\). By **Lemma 3**, \( G_{\alpha, \beta + h} > G_{\alpha, \beta} \) for all \( h > 0 \), so \( \beta^*_v \) must be decreasing as long as \( \mathcal{V}_v(\alpha) \) is non-empty. If \( \alpha_n \uparrow \alpha^*_v, \beta_n = \beta^*_v(\alpha_n) \) is decreasing and non-negative and thus must have a finite limit \( \beta^*_v \). Taking limits in \( \alpha_n + \beta_n \geq 0 \) and \( h_{\alpha_n, \beta_n} \leq v_v \) implies that \( \alpha^*_v + \beta^*_v \geq 0 \) and \( \beta^*_v \leq v_v \). Thus, \( \mathcal{V}_v(\alpha^*_v) \) is non-empty. As \( \mathcal{U}_v = \bigcup_v \mathcal{V}_v(\alpha) \), the first assertion follows. Finally, as \( \mathcal{U}_v \) is closed and convex, \( \beta^*_v \) must be continuous and concave. \( \square \)

**Theorem 6.** Let \( \nu \geq -1 \). For \( \alpha \leq \nu, \beta^*_v(\alpha) = 2(\nu + 1) - \alpha \). For \( \nu < \alpha \leq \alpha^*_v, \beta^*_v(\alpha) < 2(\nu + 1) - \alpha \).

**Proof.** We know that \( (\nu, \nu + 2) \in \mathcal{U}_v \). By **Theorem 1**, \( G_{\nu, \beta} \geq R_v \) is only possible if \( \alpha + \beta \leq 2(\nu + 1) = \nu + (\nu + 2) \) so that \( \beta^*_v(\alpha) \leq 2(\nu + 1) - \alpha \). If \( \alpha + \beta = 2(\nu + 1) \) and \( \beta > 0 \),
\[ h_{\alpha, \beta}(t) = \frac{t^3}{2\beta} + O(t^4), \quad t \to 0 \]
by Eq. (3) and comparison with Eq. (2) shows that \( h_{\alpha, \beta} \leq v_v \) is only possible if in fact \( \beta \geq \nu + 2 > 0 \), or equivalently, if \( \alpha \leq 2(\nu + 1) - (\nu + 2) = \nu \). For \( \alpha < \nu \), **Lemma 3** implies that \( G_{\alpha, 2(\nu + 1) - \alpha} > G_{\nu, \nu + 2} \geq R_v \), so that indeed \( \beta^*_v(\alpha) = 2(\nu + 1) - \alpha \). \( \square \)

Let
\[ Q_{\alpha, \beta}(s) = \beta^2 + (2(\nu + 1) - \alpha - \beta^2)s + 2(\nu + 1/2 - \alpha)s^2. \]

**Lemma 5.** Let \( \Delta = \nu_v - h_{\alpha, \beta} \). Then
\[ t \Delta'(t) = \frac{Q_{\alpha, \beta}(\sqrt{t^2 + \beta^2})}{\sqrt{t^2 + \beta^2}} + (2(\nu + 1) - \nu_v(t) - h_{\alpha, \beta}(t)) \Delta(t). \]

**Proof.** As shown in Simpson and Spector [9], \( \nu_v \) satisfies the Riccati equation \( t \nu'_v(t) = t^2 + 2(\nu + 1)v_v(t) - v_v(t)^2 \) and clearly, \( h_{\alpha, \beta}(t) = \frac{t}{\sqrt{t^2 + \beta^2}} \). Hence, as \( v^2 = h^2 + (v^2 - h^2) = h^2 + (v - h)(v + h) \),
\[ t \nu'_v(t) = t^2 + 2(\nu + 1)(\Delta(t) + h_{\alpha, \beta}(t)) - (h_{\alpha, \beta}(t)^2 + \Delta(t) + h_{\alpha, \beta}(t)) \]
\[ = t^2 + 2(\nu + 1)h_{\alpha, \beta}(t) - (h_{\alpha, \beta}(t)^2 + (2(\nu + 1) - \nu_v(t) - h_{\alpha, \beta}(t)) \Delta(t) \]
with
\[ t^2 + 2(\nu + 1)h_{\alpha, \beta}(t) - h_{\alpha, \beta}(t)^2 = t^2 + 2(\nu + 1) \left( \alpha + \sqrt{t^2 + \beta^2} \right) - \left( \alpha^2 + 2\alpha \sqrt{t^2 + \beta^2} + t^2 + \beta^2 \right) \]
\[ = 2(\nu + 1) - \alpha^2 - \beta^2 = 2(\nu + 1 - \alpha \sqrt{t^2 + \beta^2}) \]
so that
\[ t^2 + 2(\nu + 1)h_{\alpha, \beta}(t) - h_{\alpha, \beta}(t)^2 - \frac{t^2}{\sqrt{t^2 + \beta^2}} = \frac{(2(\nu + 1) - \alpha^2 - \beta^2) \sqrt{t^2 + \beta^2} + 2(\nu + 1 - \alpha)(t^2 + \beta^2) - t^2}{\sqrt{t^2 + \beta^2}} \]
\[ = \frac{Q_{\alpha, \beta}(\sqrt{t^2 + \beta^2})}{\sqrt{t^2 + \beta^2}}, \]
whence the lemma. \( \square \)
Let
\[ \alpha_v^\circ = \min(v + 1/2, 2v + 1) \]
(so that \( \alpha_v^\circ \) equals \( v + 1/2 \) for \( v \geq -1/2 \) and \( 2v + 1 \) otherwise), and for \(-1 \leq v \leq \alpha \leq \alpha_v^\circ \) let
\[ \beta_v^\circ(\alpha) = \sqrt{2v + 1 - 2\alpha} + \sqrt{2v + 1 + 2\alpha - \alpha^2} = \sqrt{2(v + 1/2 - \alpha)} + \sqrt{(\alpha + 1)(2v + 1 - \alpha)} \]
(where the second expressions shows that \( \beta_v^\circ \) is well-defined).

**Lemma 6.** Let \( v \geq -1 \). Then \( \beta_v^\circ \) is strictly concave with \( \beta_v^\circ(v) = v + 2 \), \( \beta_v^\circ(\alpha_v^\circ) \) equals \( (v + 1/2)(v + 3/2) \) if \( v \geq -1/2 \) and \( \sqrt{-2(v + 1/2)} \) if \(-1 \leq v \leq -1/2 \), and \( \alpha \mapsto \alpha + \beta_v^\circ(\alpha) \) is non-negative and decreasing.

**Proof.** The assertions about the values of \( \beta_v^\circ \) at \( v \) and \( \alpha_v^\circ \) are straightforward. If \( v = -1 \), \( \alpha_v^\circ = v \) and there is nothing left to prove. Hence, take \( v > -1 \). The second derivative of \( \alpha \mapsto \sqrt{f(\alpha)} \) is given by
\[ \frac{d^2\sqrt{f(\alpha)}}{d\alpha^2} = \frac{f''(\alpha)f(\alpha) - f'(\alpha)^2/2}{2\sqrt{f(\alpha)}}. \]
For \( f_1(\alpha) = 2(v + 1/2 - \alpha) \) and \( f_2(\alpha) = 2v + 1 + 2\alpha - \alpha^2 \) we have \( f_1'(\alpha) = -2, f_1''(\alpha) = 0, f_2'(\alpha) = 2(\alpha - v) \) and \( f_2''(\alpha) = -2 \), giving numerators \(-2\) and \(-2(2v + 1 + 2\alpha - \alpha^2) - 4(\alpha - v)^2/2 = -2(2v + 1)^2 < 0 \). Hence \( \beta_v^\circ \) is the sum of two strictly concave functions, and thus strictly concave. Clearly,
\[ \frac{d\beta_v^\circ(\alpha)}{d\alpha} = \frac{-1}{2\sqrt{2v + 1 - 2\alpha} + \sqrt{2v + 1 + 2\alpha - \alpha^2}} \]
with value \(-1 \) at \( \alpha = v \). By strict concavity, the derivative of \( \beta_v^\circ \) is decreasing, and hence less than \(-1 \) for \( \alpha > v \), so that the derivative of \( \alpha \mapsto f(\alpha) = \alpha + \beta_v^\circ(\alpha) \) is negative for \( \alpha > v \) and \( f \) is decreasing. It remains to show that \( f(\alpha_v^\circ) \geq 0 \). If \( v \geq -1/2 \), this is immediate from \( \alpha_v^\circ = v + 1/2 \geq 0 \). Otherwise, \( \alpha_v^\circ = 2v + 1 < 0 \) and \( f(\alpha_v^\circ) = 2v + 1 + \sqrt{-2(2v + 1)} \), which is non-negative as \( 0 \leq 2(v + 1) \leq 1 \).

**Theorem 7.** Let \( v \geq -1 \). Then for \( v \leq \alpha \leq \alpha_v^\circ, G_{\alpha,\beta_v^\circ(\alpha)} \geq R_v. \)

**Proof.** The proof will be based on the ideas of Simpson and Spector [9]. Suppose \( \Delta \) is sufficiently often continuously differentiable on \([0, \infty)\) with \( \Delta(0) > 0 \). Suppose that for all \( t > 0 \), \( \Delta(t) = 0 \) implies that there exists a suitable odd \( k \) such that \( \Delta^{(k)}(t) = 0 \) for \( l < k \) and \( \Delta^{(k)}(t) > 0 \). Then \( \Delta(t) \geq 0 \) for all \( t \geq 0 \), as otherwise for \( s = \inf(t > 0 : \Delta(t) = 0) \) we would have \( \Delta(s - \epsilon) = \Delta^{(k)}(s)(-\epsilon)^k/k! < 0 \) for all sufficiently small \( \epsilon > 0 \) and a suitable \( s^* \in (s - \epsilon, s) \), which is impossible.

In our case, \( \Delta = v_v - h_v, \beta = \beta_v^\circ(\alpha). \) If \( \alpha = v \), we have \( \beta = v + 2 \) and we already know for \( v \geq -1 \) that \( G_{\alpha,\beta} = G_{v,v+2} \geq R_v. \) By **Lemma 6**, \( \alpha + \beta_v^\circ(\alpha) \) is decreasing and hence maximal for \( \alpha = v \) with value \( 2(v + 1) \). Thus, for \( \alpha > v \) we have \( \alpha + \beta_v^\circ(\alpha) < 2(v + 1) \), or equivalently, \( \Delta(0) > 0 \).

Write \( s(t) = \sqrt{t^2 + \beta^2}. \) If \( \alpha = v + 1/2 \), which is only possible if \( v \geq -1/2 \), we have \( \beta = \sqrt{(v + 1/2)(v + 3/2)} \) and \( Q_{\alpha,\beta} = \beta^2 \) for all \( s \). If \( v = -1/2 \), we already know that \( R_{-1/2} = \tanh \leq G_{0,0}. \) Otherwise, \( Q_{\alpha,\beta}(s) = \beta^2 > 0 \). If \( \Delta(t) = 0 \) for some \( t > 0, \) **Lemma 5** implies that \( \Delta'(t) = \beta^2/t(s(t)) > 0 \), completing the proof for this case.

Hence, consider the case where \( v < \alpha < v + 1/2 \). Solving \( Q_{\alpha,\beta}(s) = 0 \) has discriminant
\[ (2(v + 1)\alpha - \alpha^2 - \beta^2)^2 - 8(v + 1/2 - \alpha)\beta^2 = \left(2(v + 1)\alpha - \alpha^2 - \beta^2 + 2\beta\sqrt{2v + 1 - 2\alpha}\right) \left(2(v + 1)\alpha - \alpha^2 - \beta^2 - 2\beta\sqrt{2v + 1 - 2\alpha}\right), \]
with \( \beta = \beta_v^\circ(\alpha) \) the largest root of the first factor. Hence, the discriminant vanishes, and with
\[ \sigma = -2(2(v + 1)\alpha - \beta^2)/4(v + 1/2 - \alpha) = \frac{2\sqrt{2v + 1 - 2\alpha\beta}}{4(v + 1/2 - \alpha)} = \frac{\beta}{\sqrt{2v + 1 - 2\alpha}} > 0 \]
we have \( Q_{\alpha,\beta}(s) = \gamma(s - \sigma)^2, \) where \( \gamma = 2v + 1 - 2\alpha > 0. \)

If \( \Delta(t) = 0 \) for some \( t > 0, \) **Lemma 5** implies that \( t\Delta'(t) = Q_{\alpha,\beta}(s(t))/s(t). \) If \( s(t) \neq \sigma, Q_{\alpha,\beta}(s(t)) > 0, \) and the proof is complete. Otherwise, use **Lemma 5** to write \( t\Delta'(t) = \xi(t) + \eta(t)\Delta(t), \) where
\[ \xi(t) = \gamma(s(t) - \sigma)^2/s(t) = \gamma \left(s(t) - 2\sigma + \frac{\sigma^2}{s(t)}\right) \]
so that \( \xi'(t) = \gamma(s'(t) - \sigma^2s(t)/s(t)^2) \) and
\[ \xi''(t) = \gamma \left(s''(t) - \sigma^2 \left(\frac{s'(t)^2}{s(t)^2} - 2\frac{s'(t)^2}{s(t)^3}\right)\right). \]
If \( s(t) = \sigma, \xi'(t) = 0 \) and \( \xi''(t) = 2\gamma s'(t)^2 / \sigma > 0 \). Differentiation gives \( \Delta'(t) + t \Delta''(t) = \xi'(t) + \eta'(t) \Delta(t) + \eta(t) \Delta'(t) \) and \( 2\Delta''(t) + t \Delta'''(t) = \xi''(t) + \eta''(t) \Delta(t) + 2\eta'(t) \Delta'(t) + \eta(t) \Delta''(t) \), so that if \( s(t) = \sigma, \Delta(t) = \Delta'(t) = \Delta''(t) = 0 \) and \( \Delta'''(t) = \xi''(t) / t > 0 \), and the proof is complete. \( \square \)

**Theorem 8.** Let \( v \geq -1 \). Then the elements of \( \{ G_{\alpha, \beta^v(\alpha)} : v \leq \alpha \leq \alpha^v \} \) are mutually incomparable.

**Proof.** By Lemma 6, \( \alpha \mapsto \alpha + \beta^v(\alpha) \) is decreasing, whence the result by using Lemma 3. \( \square \)

**Theorem 9.** For \( v \geq -1/2, \alpha^v \geq v + 1/2 \) and

\[
\beta^v_*(v + 1/2) = \beta^v_*(v + 1/2) = \sqrt{(v + 1/2)(v + 3/2)}.
\]
For \(-1 \leq v < -1/2, \alpha^v < v + 1/2 \).

**Proof.** Let \( \beta^v = \beta^v_*(v + 1/2) = \sqrt{(v + 1/2)(v + 3/2)} \). For arbitrary \( \beta \),

\[
G_{v+1/2, \beta} = 1 - \frac{\nu}{\nu + 1/2} + \frac{2(v + 1/2)^2 - \beta^2}{2t^2} + O(t^{-3}), \quad t \to \infty
\]
by Eq. (4) and comparison with Eq. (1) shows that \( G_{v+1/2, \beta} \geq R_v \) is only possible if

\[
2(v + 1/2)^2 - \beta^2 \geq (v + 1/2)(v - 1/2),
\]
or equivalently, if \( \beta^2 \leq 2(v + 1/2)^2 - (v + 1/2)(v - 1/2) = (v + 1/2)(v + 3/2) \). For \( v < -1/2, \) the upper bound is negative, so that \( G_{v+1/2, \beta} \geq R_v \) is impossible for all \( \beta \geq 0 \) and hence \( \alpha^v_0 < v + 1/2 \). For \( v \geq 1/2, \) the condition is equivalent to \( \beta \leq \beta^v_0 \).

By Theorem 7, \( (v + 1/2, \beta^v_0) \in U_v \), and by Theorem 1, \( \alpha \geq v + 1/2 \), so that \( \alpha^v_0 = v + 1/2 \) and \( \beta^v_*(v + 1/2) = \beta^v_0 \). \( \square \)

**Theorem 10.** Let \( v \geq -1/2 \) and \( v < \alpha < v + 1/2 \). Then there exists a unique positive \( t^v_*(\alpha) \) at which \( G_{\alpha, \beta^v(\alpha)} \) is tangent to \( R_v \). The map \( \alpha \mapsto t^v_*(\alpha) \) is continuous and increasing on \( (v, v + 1/2) \), with \( \lim_{\alpha \to v+} t^v_*(\alpha) = 0 \) and \( \lim_{\alpha \to v+1/2-} t^v_*(\alpha) = \infty \).

**Proof.** Write \( \beta^v = \beta^v_*(\alpha) \). By Theorem 6, we can find \( \delta > 0 \) such that \( \beta^v \leq 2(v + 1) - \alpha - \delta \). Using Lemma 3 and the fact that \( \sqrt{(v + 1/2)(v + 3/2)} \leq \beta^v_*(\alpha) \leq \beta^v \), we can find \( 0 < t_1 < t_2 \) such that for all \( \beta^v \leq \beta \leq \beta^v + \delta, \) \( G_{v+1/2, \beta} > R_v \) for \( 0 < t \leq t_1 \) and \( G_{v, \beta} \geq G_{v+1/2, \nu} \geq G_{v+1/2, \nu}(v + 1/2)(v + 3/2) > R_v \) for \( t \geq t_2 \). If \( \beta^v > R_v \), we have for all \( \eta \) sufficiently small that \( G_{v, \beta^v+\eta} > R_v \) for \( t_1 \leq t \leq t_2 \). Hence, \( \beta^v + \eta > R_v \) for all \( \eta > 0 \) sufficiently small, which contradicts the maximality of \( \beta^v \). Thus, there must be at least one \( t > 0 \) such that \( G_{v, \beta^v}(t) = R_v \), and clearly, the derivatives must agree at \( t \) as otherwise \( \beta^v \) could not be an upper bound for \( R_v \). Equivalently, \( h_{v, \beta} \) must be tangent to \( \nu_v \) at \( t \). By Lemma 5, this is the case iff \( t \) solves \( Q_{v, \beta^v} \left( \sqrt{t^2 + \beta^v_*} \right) = 0 \), from which we infer that \( t = t^v_*(\alpha) \) is uniquely determined and continuous as a function of \( \alpha \). The limits for \( \alpha \to v \) from the right and \( \alpha \to v + 1/2 \) from the left are obvious. To show that \( t^v_*(\alpha) \) is increasing, it suffices to show that it is injective. Hence, let \( v < \alpha_1 < \alpha_2 < v + 1/2 \) and suppose that \( t^v_*(\alpha_1) = t^v_*(\alpha_2) = t^v \). Then \( \beta^v_*(\alpha_1) = \beta^v_*(\alpha_2) = \beta^v \), the \( h_{v, \beta^v_*(\alpha_1)} \) must have the same value and derivative at \( t^v \), so that

\[
\frac{\sqrt{t^v + \beta^v_*}}{\sqrt{t^v + \beta^v_*^2}} = \frac{t^v}{\sqrt{t^v + \beta^v_*^2}},
\]
and hence \( \beta^v_1 = \beta^v_2 \), which is impossible as \( \beta^v \) is decreasing by Theorem 5. \( \square \)

**Theorem 11.** Let \( v \geq -1/2 \). Then \( \{ G_{\alpha, \beta^v(\alpha)} : v \leq \alpha \leq v + 1/2 \} \) are the minimal elements of the family \( g_{U_v} \) of upper Amos-type bounds for \( R_v \), and

\[
R_v = \min \{ G_{\alpha, \beta^v(\alpha)} : v \leq \alpha \leq v + 1/2 \}.
\]

**Proof.** Let \( t > 0 \). By Theorem 10, there exists a unique \( v < \alpha < v + 1/2 \) so that \( t^v_*(\alpha) = t \) and hence \( R_v(t) = G_{\alpha, \beta^v(\alpha)}(t) \), proving the second assertion. Let \( v \leq \alpha_1 < \alpha_2 \leq v + 1/2 \) and \( \beta^v_1 = \beta^v_2 = \beta^v_*(\alpha_1) \). If \( \alpha_1 = v \), Theorem 6 shows that \( 2(v + 1) = \alpha_1 + \beta^v_1 > \alpha_2 + \beta^v_2 \). If \( \alpha_1 > v \) and \( \alpha_1 + \beta^v_1 \leq \alpha_2 + \beta^v_2 \), Lemma 3 implies that \( R_v \leq G_{\alpha_2, \beta^v_2} < G_{\alpha_1, \beta^v_1} \), which is impossible as by Theorem 10, \( G_{\alpha_1, \beta^v_1} \) must be the only tangent to \( R_v \) at \( t^v_*(\alpha_1) \). Thus we always have \( \alpha_1 + \beta^v_1 \geq \alpha_2 + \beta^v_2 \), and again by Lemma 3, there always exists \( t = t(\alpha_1, \alpha_2) \) such that \( G_{\alpha_1, \beta^v_1}(s) < G_{\alpha_2, \beta^v_2}(s) \) for \( 0 < s < t \) and \( G_{\alpha_1, \beta^v_1}(s) > G_{\alpha_2, \beta^v_2}(s) \) for \( s > t \). As \( G_{\alpha, \beta^v(\alpha)} > G_{v+1/2} = G_{v, \beta^v(\alpha)} \) for \( \alpha < v \) and trivially \( G_{\alpha, \beta} \geq G_{\alpha, \beta^v(\alpha)} \) provided that \( (\alpha, \beta) \in U_v \), the first assertion follows, and the proof is complete. \( \square \)
Finally, let us consider the cases where $\nu = -k$ is a negative integer. As readily seen from the series expansion, $I_{-k} = I_k$, and hence $R_{-k} = L_{-k+1}/L_{-k} = l_{k-1}/l_k = 1/R_{k-1}.$

**Theorem 12.** If $k$ is a positive integer,
\[ \mathcal{U}_{-k} = \{(-\beta, \beta) : \beta \geq k\} \]
and $G_{-k,k}$ is the minimum of the family $g_{\mathcal{U}_k}$ of upper Amos-type bounds for $R_{-k}$.

**Proof.** As $R_{-k} > 0$ and has a pole at $t = 0$, the same must be true for upper bounds $G_{\alpha, \beta}$ of $R_{-k}$, implying that necessarily $\alpha + \beta = 0$. As
\[
G_{-\beta, \beta}(t) = \frac{t}{\sqrt{t^2 + \beta_1^2 - \beta}} = \frac{t(\sqrt{t^2 + \beta_1^2 + \beta})}{(t^2 + \beta_1^2 - \beta^2)} = \frac{\sqrt{t^2 + \beta_1^2 + \beta}}{t} = \frac{1}{G_{\beta, \beta}(t)},
\]
we have $1/G_{\beta, \beta} = G_{-\beta, \beta} \geq R_{-k} = 1/R_{k-1}$ iff $R_{k-1} \geq G_{\beta, \beta}$, i.e., $(\beta, \beta) \in \mathcal{L}_{k-1}$. From the characterization of $\mathcal{L}_{\nu}$ for $\nu \geq -1$ (Theorem 3), this is possible iff $\beta \geq k - 1/2$ and $2\beta \geq 2k$, or equivalently, $\beta \geq k$. □

**Theorem 13.** If $k$ is a positive integer,
\[ \mathcal{L}_{-k} = \{(\alpha, \beta) : \alpha \geq -(k - 1/2), \alpha + \beta \geq 0, \beta \geq 0\} \]
and $G_{-(k-1)/2,k-1/2}$ is the maximum of the family $g_{\mathcal{L}_{\nu}}$ of lower Amos-type bounds for $R_{-k}$.

**Proof.** For lower bounds $G_{\alpha, \beta}$ of $R_{-k}$, we must have $\alpha + \beta \geq 0$ by the usual arguments, and Theorem 1 implies that necessarily $\alpha \geq -k + 1/2$. On the other hand, we also know that $R_{k-1} \leq G_{k-1/2,k-1/2}$, or equivalently, $G_{-(k-1)/2,k-1/2} \leq R_{-k}$, and the proof is complete. □

Note that for $k = 1$, we already know by Theorem 3 that $G_{1+1/2,-1+1/2} = G_{-1/2,1/2}$ is the greatest lower bound for $R_{-1}$, and Theorem 6 yields that $\beta^{\flat-1}_{1,1} = 1$, so that $G_{-1,1}$ is the least upper bound for $R_{-1}$ with $\alpha = -1$.

5. Summary and conclusions

In this paper, we systematically investigate lower and upper Amos-type bounds for $R_v = I_{v+1}/I_v$ on the positive reals when $R_v$ is positive, or equivalently, when $\nu \geq -1$ or $\nu$ is a negative integer.

For $\nu \geq -1$, the set $\mathcal{L}_{\nu}$ of all $(\alpha, \beta)$ giving lower bounds $G_{\alpha, \beta} \leq R_v$ has a simple explicit description, and $G_{\nu+1/2,\nu+3/2}$ is the maximum of the family $g_{\mathcal{L}_{\nu}}$ of lower Amos-type bounds for $R_v$ (Theorem 3).

For $\nu \geq -1$, the set $\mathcal{U}_{\nu}$ of all $(\alpha, \beta)$ giving upper bounds $G_{\alpha, \beta} \geq R_v$ is of the form $\{(-\alpha, \alpha) : \alpha \leq \alpha^*_v, \max(0, -\alpha) \leq \beta \leq \beta^*_v(\alpha)\}$, where $\nu \leq \alpha^*_v \leq \nu + 1/2$ and $\beta^*_v(\alpha)$ is continuous, decreasing and concave (Theorem 5), with $\beta^*_v(\nu) = \nu + 2$ and $\alpha + \beta^*_v(\alpha) < 2(\nu + 1)$ for $\alpha > \nu$ (Theorem 6). If $\nu \geq -1/2, \alpha^*_v = \nu + 1/2$ and $\beta^*_v(\nu + 1/2) = (\nu + 1/2)(\nu + 3/2)$ by Theorem 9, and the upper bounds in the family $\{G_{\nu, \nu^*}(\alpha), \nu \leq \alpha \leq \nu + 1/2\}$ are tangent to $R_v$ in exactly one point $t^*_v(\alpha)$ (Theorem 10, taking $t^*_v(\nu) = 0$ and $t^*_v(\nu + 1/2) = \infty$), and the minimal elements of the family $g_{\mathcal{U}_{\nu}}$ of upper Amos-type bounds for $R_v$, with $R_v$ as their lower envelope (Theorem 11).

Thus, for $\nu \geq -1$, the pointwise maximum over all lower Amos-type bounds equals $G_{\nu+1/2,\nu+3/2} < R_v$, and hence is always smaller than $R_v$. On the other hand, for $\nu \geq -1/2$, the pointwise minimum over all upper Amos-type bounds equals $R_v$.

For $\nu \geq -1$ and $\nu < \alpha < \alpha^*_v = \min(\nu + 1/2, 2\nu + 1)$, Theorems 7 and 8 establish a family $\{G_{\alpha, \nu^*_\alpha}(\alpha), \nu \leq \alpha \leq \alpha^*_v\}$ of explicitly computable, mutually incomparable upper bounds for $R_v$ with $\beta^*_v(\nu) = \beta^*_v(\nu) = \nu + 2$. For $\nu < \alpha < \alpha^*_v$, these bounds are new. For $\nu \geq -1/2, \alpha^*_v = \alpha^*_v = \nu + 1/2$ and $\beta^*_v(\nu + 1/2) = \beta^*_v(\nu + 1/2)$, and Theorem 7 extends the range of the bound $G_{\nu+1/2,\nu+1/2}$ for $\nu \geq -1/2$, given in Simpson and Spector [9] from $\nu \geq 0$ to $\nu \geq -1/2$, and for $-1/2 < \nu < 0$ dominates $G_{\nu+1/2,\nu+1/2}$ as the best previously available upper bound with $\alpha = \nu + 1/2$ (and hence first order exact as $t \to \infty$).

Finally, for the cases where $\nu = -k$ is a negative integer, Theorems 12 and 13 give explicit characterizations of $\mathcal{U}_{-k}$ and $\mathcal{L}_{-k}$, and establish $G_{-k,k}$ and $G_{-(k-1)/2,k-1/2}$ as the least upper and greatest lower Amos-type bounds for $R_{-k}$, respectively.

For $-1 \leq \nu < -1/2$, the value of $\alpha^*_v$ is not known; the results in this paper imply that $\alpha^*_v \leq \alpha^*_v < \nu + 1/2$. It is also not known whether in this case $R_v$ can be obtained as the lower envelope of all upper Amos-type bounds. For $\nu = -1$, this is certainly not the case (as $G_{-1,1}$ is the uniformly smallest upper bound). Hence the range $-1 < \nu < -1/2$ deserves further investigation.

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References