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Amos-type bounds for modified Bessel function ratios

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ABSTRACT

We systematically investigate lower and upper bounds for the modified Bessel function ratio $R_\nu = I_{\nu+1}/I_\nu$ by functions of the form $G_{\alpha,\beta}(t) = t/(\alpha+\sqrt{t^2+\beta^2})$ in case R_ν is positive for all $t > 0$, or equivalently, where $\nu \geq -1$ or ν is a negative integer. For $\nu \geq -1$, we give an explicit description of the set of lower bounds and show that it has a greatest element. We also characterize the set of upper bounds and its minimal elements. If $\nu \geq -1/2$, the minimal elements are tangent to R_ν in exactly one point $0 \leq t \leq \infty$, and have R_ν as their lower envelope. We also provide a new family of explicitly computable upper bounds. Finally, if ν is a negative integer, we explicitly describe the sets of lower and upper bounds, and give their greatest and least elements, respectively.

1. Introduction

Let I_ν be the modified Bessel function of order ν, and R_ν the (modified) Bessel function ratio $R_\nu(t) = I_{\nu+1}(t)/I_\nu(t)$. These ratios are of great importance in a variety of application areas, including statistics [e.g.,7] and numerical analysis [e.g.,1], either directly or through the fact that by the well-known recurrence relations for modified Bessel functions,

$$\log(I_\nu)'(t) = \frac{I_\nu'(t)}{I_\nu(t)} = \frac{I_{\nu+1}(t) + (\nu/t)I_\nu(t)}{I_\nu(t)} = R_\nu(t) + \frac{\nu}{t}$$

from which by integration and taking limits,

$$\log(I_\nu)(t) = \int_0^t R_\nu(s)\,ds + \nu\log(t/2) - \log(\Gamma(\nu+1)).$$

For functions f and g defined on the positive reals, write $f \leq g$ iff $f(t) \leq g(t)$ for all $t > 0$, with $f < g$ defined analogously. If neither $f \leq g$ nor $g \leq f$, we say that f and g are incomparable. Let $\mathcal{G}$ be a family of functions on the positive reals and $f \in \mathcal{G}$. We say that f is the least element (minimum) of $\mathcal{G}$ iff $f \leq g$ for all $g \in \mathcal{G}$, and that f is a minimal element of $\mathcal{G}$ iff there is no $g \in \mathcal{G}$ for which $f > g$, with the greatest element (maximum) and maximal elements of $\mathcal{G}$ defined analogously.

Let

$$G_{\alpha,\beta}(t) = \frac{t}{\alpha + \sqrt{t^2+\beta^2}},$$

where in what follows we always (without loss of generality) take $\beta \geq 0$. For $\nu \geq 0$, Eqs. (9), (11) and (16) in Amos [1] show that

$$\max(G_{\nu+1,\nu+1}, G_{\nu+1/2,\nu+3/2}) \leq R_\nu \leq \min(G_{\nu,\nu}, G_{\nu,\nu+2}, G_{\nu+1/2,\nu+1/2}).$$

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Such "Amos-type" bounds were re-established and extended in several publications (see Section 3 for details). These bounds are very attractive because they allow both for explicit inversion and integration. Thus, Amos-type bounds yield bounds (and approximations) also for R_ν^{-1} and the antiderivate of R_ν (equivalently, I_ν and its logarithm).

Let

$$\mathcal{L}_\nu = \{(\alpha, \beta) : G_{\alpha,\beta} \le R_\nu\}, \qquad \mathcal{U}_\nu = \{(\alpha, \beta) : G_{\alpha,\beta} \ge R_\nu\}$$

be the set of all (α, β) for which $G_{\alpha,\beta}$ is a lower/upper Amos-type bound for R_ν, and write

$$\mathcal{G}_{\mathcal{L}_\nu} = \{G_{\alpha,\beta} : (\alpha, \beta) \in \mathcal{L}_\nu\}, \qquad \mathcal{G}_{\mathcal{U}_\nu} = \{G_{\alpha,\beta} : (\alpha, \beta) \in \mathcal{U}_\nu\},$$

for the corresponding families of lower/upper Amos-type bounds for R_ν.

In this paper, we investigate the structure of $\mathcal{G}_{\mathcal{L}_\nu}$ and $\mathcal{G}_{\mathcal{U}_\nu}$ under the condition that $R_\nu > 0$, or equivalently, $\nu \ge -1$ or ν a negative integer.

2. Preliminaries

Let

$$v_\nu(t) = tI_\nu(t)/I_{\nu+1}(t) = t/R_\nu(t)$$

and

$$h_{\alpha,\beta}(t) = \alpha + \sqrt{t^2 + \beta^2}$$

so that $G_{\alpha,\beta}(t) = t/h_{\alpha,\beta}(t)$.

Using, e.g., Watson [10, Formula 3.7.2],

$$R_\nu(t) = \frac{t}{2} \frac{\sum_{n=0}^{\infty} t^{2n} \Big/ (4^n n! \Gamma(n+\nu+2))}{\sum_{n=0}^{\infty} t^{2n} \Big/ (4^n n! \Gamma(n+\nu+1))}.$$

If $\nu \ge -1$, all coefficients in the numerator and denominator series are non-negative and eventually positive, and hence $R_\nu > 0$. If ν is a negative integer, the same is true; otherwise, $\lim_{t\to 0} v_\nu(t) = 2\Gamma(\nu+2)/\Gamma(\nu+1) = 2(\nu+1)$ which is negative if $\nu < -1$, and hence $R_\nu(t) < 0$ for all sufficiently small positive t.

Using the asymptotic expansion of I_ν for large argument [10, e.g., Formula 7.23.2], one can show that for arbitrary ν,

$$R_\nu(t) = 1 - \frac{\nu + 1/2}{t} + \frac{\nu^2 - 1/4}{2t^2} + O(1/t^3), \quad t \to \infty, \tag{1}$$

see also Schou [7, Eq. (6), assuming $\nu \ge 0$].

As $h_{\alpha,\beta}$ is increasing with $h_{\alpha,\beta}(0) = \alpha + \beta$, we have $G_{\alpha,\beta} > 0$ iff $\alpha + \beta \ge 0$. Hence, when $\nu \ge -1$ or ν is a negative integer and $\alpha + \beta \ge 0$, $G_{\alpha,\beta}$ is a (strict) upper or lower bound for R_ν if and only if $h_{\alpha,\beta}$ is a (strict) lower or upper bound for v_ν, respectively.

Lemma 1. *For $\nu \ge -1$,*

$$v_\nu(t) = 2(\nu+1) + \frac{t^2}{2(\nu+2)} + O(t^4), \quad t \to 0. \tag{2}$$

Proof. More generally, if ν is not a negative integer,

$$\begin{aligned} v_\nu(t) &= t \frac{(t/2)^\nu \left(\frac{1}{\Gamma(\nu+1)} + \frac{t^2/4}{\Gamma(\nu+2)} + O(t^4)\right)}{(t/2)^{\nu+1} \left(\frac{1}{\Gamma(\nu+2)} + \frac{t^2/4}{\Gamma(\nu+3)} + O(t^4)\right)} = 2 \frac{(\nu+1) + \frac{t^2}{4} + O(t^4)}{1 + \frac{t^2}{4(\nu+2)} + O(t^4)} \\ &= 2(\nu+1) + \frac{t^2}{2(\nu+2)} + O(t^4), \quad t \to 0. \end{aligned}$$

If $\nu = -k$ is a negative integer, $1/\Gamma(\nu+n+1)$ vanishes for n from 0 to $k-1$, and hence

$$\begin{aligned} v_{-k}(t) &= 2 \frac{\sum_{n=k}^{\infty} t^{2n} \Big/ (4^n n! \Gamma(n-k+1))}{\sum_{n=k-1}^{\infty} t^{2n} \Big/ (4^n n! \Gamma(n-k+2))} \\ &= 2 \frac{t^{2k}/(4^k k!)}{t^{2(k-1)}/(4^{k-1}(k-1)!)} \frac{1 + \frac{t^2}{4(k+1)} + O(t^4)}{1 + \frac{t^2}{4k} + O(t^4)} = \frac{t^2}{2k} + O(t^4), \quad t \to 0. \end{aligned}$$

As for $\nu = -k = -1$ we have $2(\nu+1) = 0$ and $\nu+2 = 1 = k$, we can combine the two expansions to obtain the lemma. $\square$

Lemma 2. *If $\beta > 0$,*

$$h_{\alpha,\beta}(t) = (\alpha+\beta) + \frac{t^2}{2\beta} + O(t^4), \quad t \to 0. \tag{3}$$

For arbitrary α and $\beta \geq 0$,

$$G_{\alpha,\beta}(t) = 1 - \frac{\alpha}{t} + \frac{2\alpha^2-\beta^2}{2t^2} + O(t^{-3}), \quad t \to \infty. \tag{4}$$

Proof. If $\beta > 0$, then

$$\sqrt{t^2+\beta^2} = \beta\sqrt{1+(t/\beta)^2} = \beta\left(1 + \frac{t^2}{2\beta^2} + O(t^4)\right) = \beta + \frac{t^2}{2\beta} + O(t^4)$$

for $t \to 0$, whence Eq. (3) by adding α.

As $t \to \infty$, $\sqrt{1+\beta^2/t^2} = 1 + \beta^2/(2t^2) + O(t^{-4})$ and thus

$$\begin{aligned} G_{\alpha,\beta}(t) &= \frac{1}{\alpha/t + \sqrt{1+\beta^2/t^2}} = \frac{1}{1+\alpha/t+\beta^2/(2t^2)+O(t^{-4})} \\ &= 1 - \frac{\alpha}{t} + \frac{2\alpha^2-\beta^2}{2t^2} + O(t^{-3}), \quad t \to \infty. \quad \square \end{aligned}$$

Theorem 1. *For arbitrary ν, $G_{\alpha,\beta} \leq R_\nu$ or $G_{\alpha,\beta} \geq R_\nu$ are only possible when $\alpha \geq \nu + 1/2$ or $\alpha \leq \nu + 1/2$, respectively. If $\nu \geq -1$, then $G_{\alpha,\beta} \leq R_\nu$ or $G_{\alpha,\beta} \geq R_\nu$ are only possible when $\alpha+\beta \geq 2(\nu+1)$ or $0 \leq \alpha+\beta \leq 2(\nu+1)$, respectively.*

Proof. The first assertion is immediate by comparing the expansions of R_ν and $G_{\alpha,\beta}$ for $t \to \infty$. If $\alpha+\beta < 0$, $h_{\alpha,\beta}$ has a unique zero $t > 0$, and $G_{\alpha,\beta}$ changes from $-\infty$ to ∞ at t. If $\nu \geq -1$, $R_\nu > 0$, so upper and lower $G_{\alpha,\beta}$ bounds necessarily must have $\alpha+\beta \geq 0$. The second assertion now follows by comparing the values of v_ν and $h_{\alpha,\beta}$ at $t = 0$. $\square$

Lemma 3. *Let $\beta_1 < \beta_2$ and $\min(\alpha_1+\beta_1, \alpha_2+\beta_2) \geq 0$. Then $G_{\alpha_1,\beta_1} < G_{\alpha_2,\beta_2}$ iff $\alpha_1+\beta_1 \geq \alpha_2+\beta_2$, and $G_{\alpha_1,\beta_1} > G_{\alpha_2,\beta_2}$ iff $\alpha_1 \leq \alpha_2$. Otherwise, if $\alpha_1 > \alpha_2$ and $\alpha_1+\beta_1 < \alpha_2+\beta_2$ and*

$$t = \frac{\sqrt{((\beta_2-\beta_1)^2-(\alpha_1-\alpha_2)^2)((\beta_2+\beta_1)^2-(\alpha_1-\alpha_2)^2)}}{2(\alpha_1-\alpha_2)},$$

$G_{\alpha_1,\beta_1}(s) > G_{\alpha_2,\beta_2}(s)$ for $0 < s < t$ and $G_{\alpha_1,\beta_1}(s) < G_{\alpha_2,\beta_2}(s)$ for $s > t$.

Proof. Consider $\Delta = h_{\alpha_1,\beta_1} - h_{\alpha_2,\beta_2}$. Then $\Delta(0) = (\alpha_1+\beta_1) - (\alpha_2+\beta_2)$ and as

$$\sqrt{t^2+\beta_1^2} - \sqrt{t^2+\beta_2^2} = \frac{(t^2+\beta_1^2)-(t^2+\beta_2^2)}{\sqrt{t^2+\beta_1^2}+\sqrt{t^2+\beta_2^2}} = \frac{\beta_1^2-\beta_2^2}{\sqrt{t^2+\beta_1^2}+\sqrt{t^2+\beta_2^2}} \to 0$$

as $t \to \infty$, $\Delta(t) \to \Delta(\infty) = \alpha_1 - \alpha_2$ as $t \to \infty$. As

$$\Delta'(t) = \frac{t}{\sqrt{t^2+\beta_1^2}} - \frac{t}{\sqrt{t^2+\beta_2^2}},$$

if $\beta_1 < \beta_2$ we have $\Delta' > 0$ and hence $\Delta > 0$ iff $\Delta(0) \geq 0$, and $\Delta < 0$ iff $\Delta(\infty) \leq 0$. As $\min(\alpha_1+\beta_1, \alpha_2+\beta_2) \geq 0$, $G_{\alpha_1,\beta_1} < G_{\alpha_2,\beta_2}$ (or $>$) iff $\Delta > 0$ (or $<$). Otherwise, i.e., iff $\alpha_1 > \alpha_2$ and $\alpha_1+\beta_1 < \alpha_2+\beta_2$, Δ has a unique zero t^* in $(0,\infty)$, which can be determined as follows. Let $u = \sqrt{t^2+\beta_1^2} > \beta_1$ so that $t = \sqrt{u^2-\beta_1^2}$ and $t^2+\beta_2^2 = u^2 + (\beta_2^2-\beta_1^2)$, and $\Delta(t) = 0$ iff

$$\alpha_1 + u - \alpha_2 = \sqrt{u^2 + (\beta_2^2-\beta_1^2)}.$$

Taking squares,

$$u^2 + 2(\alpha_1-\alpha_2)u + (\alpha_1-\alpha_2)^2 = u^2 + (\beta_2^2-\beta_1^2)$$

from which

$$u = \frac{\beta_2^2-\beta_1^2}{2(\alpha_1-\alpha_2)} - \frac{\alpha_1-\alpha_2}{2}.$$

Then

$$u-\beta_1=\frac{(\beta_2^2-\beta_1^2)-(\alpha_1-\alpha_2)^2}{2(\alpha_1-\alpha_2)}-\beta_1=\frac{(\beta_2-\beta_1-\alpha_1+\alpha_2)(\beta_2+\beta_1+\alpha_1-\alpha_2)}{2(\alpha_1-\alpha_2)}.$$

The numerator equals $((\alpha_2+\beta_2)-(\alpha_1+\beta_1))((\alpha_1-\alpha_2)+(\beta_1+\beta_2))>0$ so that indeed $u>\beta_1$. Similarly,

$$u+\beta_1=\frac{\beta_2^2-\beta_1^2+2\beta_1(\alpha_1-\alpha_2)-(\alpha_1-\alpha_2)^2}{2(\alpha_1-\alpha_2)}=\frac{(\beta_2+\beta_1-\alpha_1+\alpha_2)(\beta_2-\beta_1+\alpha_1-\alpha_2)}{2(\alpha_1-\alpha_2)}$$

so that with $t^2=u^2-\beta_1^2=(u-\beta_1)(u+\beta_1)$ we indeed obtain

$$t=\frac{\sqrt{((\beta_2-\beta_1)^2-(\alpha_1-\alpha_2)^2)((\beta_2+\beta_1)^2-(\alpha_1-\alpha_2)^2)}}{2(\alpha_1-\alpha_2)}$$

for the unique solution of $\Delta(t)=0$ (and equivalently $G_{\alpha_1,\beta_1}(t)=G_{\alpha_2,\beta_2}(t)$) on $(0,\infty)$. Clearly, $\Delta(s)<0$ for $0\le s<t$ and $\Delta(s)>0$ for $s>t$, so that $G_{\alpha_1,\beta_1}(s)>G_{\alpha_2,\beta_2}(s)$ for $0<s<t$ and $G_{\alpha_1,\beta_1}(s)<G_{\alpha_2,\beta_2}(s)$ for $s>t$, and the proof is complete. □

Lemma 4. *Suppose the quadratic polynomial $Q(t)=t^2+\gamma t+\delta$ has two real zeros $t_1\le t_2$. Then $Q(t)<0$ iff $t_1<t<t_2$.*

Proof. Trivial, as $Q(t)=(t-t_1)(t-t_2)$. □

3. Previous work

Amos [1] gives the bounds

$$G_{\nu+1/2,\nu+3/2}\le R_\nu\le G_{\nu+1/2,\nu+1/2},\quad \nu\ge 0$$

(Eq. (16)) and

$$G_{\nu+1,\nu+1}\le R_\nu\le G_{\nu,\nu+2}\le G_{\nu,\nu},\quad \nu\ge 0$$

(Eqs. (9) and (11)). Using Lemma 3 with $\beta_1=\nu+1<\nu+3/2=\beta_2$ and $\alpha_1+\beta_1=2\nu+2=\alpha_2+\beta_2$ we see that the first lower bound is uniformly better (larger) than the second one, whereas again with Lemma 3, neither of the upper bounds $G_{\nu+1/2,\nu+1/2}$ and $G_{\nu,\nu+2}$ is uniformly better (smaller) than the other: in fact, with $\alpha_1-\alpha_2=1/2$, $\beta_2-\beta_1=3/2$ and $\beta_2+\beta_1=2\nu+5/2$, we get

$$t=\frac{\sqrt{(9/4-1/4)(4\nu^2+10\nu+25/4-1/4)}}{2\cdot(1/2)}=2\sqrt{(\nu+1)(2\nu+3)},$$

so that $G_{\nu,\nu+2}(s)<G_{\nu+1/2,\nu+1/2}(s)$ for $0<s<t$ and $G_{\nu+1/2,\nu+1/2}(s)<G_{\nu,\nu+2}(s)$ for $s>t$.

Nåsell [5] gives rational bounds for R_ν, and notes (p. 8) that the Amos-type bounds $G_{\nu+1/2,\nu+3/2}<R_\nu$ and $R_\nu<G_{\nu+1/2,\nu+1/2}$ are valid for $\nu>-1$ and $\nu>-1/2$, respectively. But trivially $R_{-1/2}=\tanh<1=G_{0,0}$, so that the upper bound is in fact valid for $\nu\ge-1/2$.

Simpson and Spector [9, Theorem 2] show that

$$v_\nu(t)^2-(2\nu+1)v_\nu(t)-(t^2+\nu+1/2)>0,\quad t>0,\ \nu\ge 0.$$

As the quadratic function $Q(s)=s^2-(2\nu+1)s-(t^2+\nu+1/2)$ has zeros

$$\nu+1/2\pm\sqrt{(\nu+1/2)^2+(t^2+\nu+1/2)}=\nu+1/2\pm\sqrt{t^2+(\nu+1/2)(\nu+3/2)},$$

Lemma 4 implies that $v_\nu(t)>\nu+1/2+\sqrt{t^2+(\nu+1/2)(\nu+3/2)}$ and hence

$$R_\nu<G_{\nu+1/2,\sqrt{(\nu+1/2)(\nu+3/2)}},\quad \nu\ge 0.$$

Using Lemma 3, we see that this bound is uniformly better than the Amos-type bound $G_{\nu+1/2,\nu+1/2}$. To compare with $G_{\nu,\nu+2}$, note that

$$((\beta_2-\beta_1)^2-(\alpha_1-\alpha_2)^2)((\beta_2+\beta_1)^2-(\alpha_1-\alpha_2)^2)=(\beta_2^2-\beta_1^2)^2-2(\beta_2^2+\beta_1^2)(\alpha_1-\alpha_2)^2+(\alpha_1-\alpha_2)^4.$$

Thus, using Lemma 3 with $\alpha_1=\nu+1/2$, $\beta_1=\sqrt{(\nu+1/2)(\nu+3/2)}$, $\alpha_2=\nu$ and $\beta_2=\nu+2$, we get $\alpha_1-\alpha_2=1/2$, $\beta_2^2-\beta_1^2=2\nu+13/4$, $\beta_2^2+\beta_1^2=2\nu^2+6\nu+19/4$ and

$$t=\sqrt{(2\nu+13/4)^2-2(2\nu^2+6\nu+19/4)/4+1/16}=\sqrt{3\nu^2+10\nu+33/4}=\sqrt{(3\nu+11/2)(\nu+3/2)},$$

and therefore $G_{\nu+1/2,\sqrt{(\nu+1/2)(\nu+3/2)}}(s)<G_{\nu,\nu+2}(s)$ for $s>t$, and $G_{\nu,\nu+2}(s)<G_{\nu+1/2,\sqrt{(\nu+1/2)(\nu+3/2)}}(s)$ for $0<s<t$.

Neuman [6, Proposition 5] shows that

$$v_\nu^2(t) - (2\nu+1)v_\nu(t) - (t^2+\nu+1/2) < \nu+3/2, \quad t>0,\ \nu>-3/2.$$

As the quadratic function $Q(s) = s^2 - (2\nu+1)s - (t^2 + 2(\nu+1))$ has zeros

$$\nu + 1/2 \pm \sqrt{(\nu+1/2)^2 + t^2 + 2(\nu+1)} = \nu + 1/2 \pm \sqrt{t^2 + (\nu+3/2)^2},$$

Lemma 4 implies that $v_\nu(t) < \nu + 1/2 + \sqrt{t^2+(\nu+3/2)^2}$ for $t > 0$ and $\nu > -3/2$. If $\nu \geq -1$, $v_\nu > 0$ and hence $R_\nu > G_{\nu+1/2,\nu+3/2}$.

Yuan and Kalbfleisch [11, Eq. (A.5)] show that

$$G_{\nu+1,\nu+1} \leq R_\nu \leq G_{\nu,\nu}, \quad \nu > -1.$$

Baricz and Neuman [2, Theorems 2.1 and 2.2] show that if $a > 1$ and $b = 1/(4\log(a))$, then

$$v_\nu(t)^2 - (2\nu+1)v_\nu(t) - t^2 < 2(\nu+1), \quad 0 < t \leq 2b,\ \nu \geq b-2$$

and that

$$v_\nu(t)^2 - 2\nu v_\nu(t) - t^2 > 4(\nu+1), \quad t>0,\ \nu>-2$$

(the reference uses $p-1$ for ν). The former extends the earlier result of Neuman [6] when $\nu \leq -3/2$, in which case the bounds are not valid for all $t > 0$. As $s \mapsto Q(s) = s^2 - 2\nu s - (t^2 + 4(\nu+1))$ has zeros

$$\nu \pm \sqrt{\nu^2 + t^2 + 4(\nu+1)} = \nu \pm \sqrt{t^2 + (\nu+2)^2},$$

Lemma 4 yields that for $\nu \geq -1$, the latter is equivalent to $R_\nu < G_{\nu,\nu+2}$, extending the previously established ν range for this bound.

Laforgia and Natalini [4, Theorem 1.1] show that

$$\frac{-\nu + \sqrt{t^2+\nu^2}}{t} < \frac{I_\nu(t)}{I_{\nu-1}(t)}, \quad t>0,\ \nu \geq 0$$

(the condition that $t > 0$ is not stated explicitly in the theorem, but given in Eq. (1.8) of the reference used in the proof). As

$$\frac{\sqrt{t^2+\nu^2}-\nu}{t} = \frac{(t^2+\nu^2)-\nu^2}{t\left(\sqrt{t^2+\nu^2}+\nu\right)} = \frac{t}{\nu+\sqrt{t^2+\nu^2}} = G_{\nu,\nu}(t),$$

the result is equivalent to

$$R_\nu > G_{\nu+1,\nu+1}, \quad \nu \geq -1,$$

which is weaker than the $R_\nu > G_{\nu+1/2,\nu+3/2}$ bound.

Segura [8, Theorem 3] shows that

$$\frac{I_{\nu+1/2}(t)}{I_{\nu-1/2}(t)} < \frac{t}{\nu+\sqrt{t^2+\nu^2}}, \quad t>0,\ \nu\geq 0$$

or equivalently, $R_\nu < G_{\nu+1/2,\nu+1/2}$ for $\nu \geq -1/2$. For $r_\nu(t) = I_\nu(t)/(tI_{\nu-1}(t)) = R_{\nu-1}(t)/t$, Segura [8, Eqs. (22) and (61)] also shows that for $t > 0$ and $\nu \geq 0$,

$$\frac{1}{(\nu-1/2)+\sqrt{t^2+(\nu+1/2)^2}} < r_\nu(t) < \frac{1}{\nu+\sqrt{\nu^2+t^2\nu/(\nu+1)}}.$$

Clearly, the lower bound is equivalent to $R_\nu > G_{\nu+1/2,\nu+3/2}$ for $\nu \geq -1$, and the upper bound to

$$R_\nu(t) < \frac{t}{\nu+1+\sqrt{(\nu+1)^2+t^2(\nu+1)/(\nu+2)}}$$

for $t > 0$ and $\nu \geq -1$, which is weaker than the upper bound $R_\nu < G_{\nu,\nu+2}$.

Kokologiannaki [3, Theorem 2.1] shows that for $f_\nu(t) = I_{\nu+1}(t)/(tI_\nu(t)) = R_\nu(t)/t$,

$$-\frac{\nu+1}{t^2} + \sqrt{\frac{(\nu+1)^2}{t^4}+\frac{1}{t^2}} < f_\nu(t) < -\frac{\nu+1}{t^2} + \sqrt{\frac{(\nu+1)^2}{t^4}+\frac{1}{t^2}+\frac{1}{4(\nu+1)^2(\nu+2)}}$$

for $t > 0$ and $\nu > -1$. As

$$-\frac{\nu+1}{t} + \sqrt{\frac{(\nu+1)^2}{t^2}+1} = \frac{\sqrt{t^2+(\nu+1)^2}-(\nu+1)}{t},$$

the lower bound again is equivalent to $R_\nu > G_{\nu+1,\nu+1}$ for $\nu > -1$. Write $U_K(t)$ for the above upper bound and $\gamma = 1/(4(\nu+1)^2(\nu+2))$. $U_K(t)$ is the larger root of the quadratic polynomial

$$s \mapsto Q(s;t) = s^2 + \frac{2(\nu+1)}{t^2}s - \frac{1}{t^2} - \gamma,$$

so by Lemma 4, for any function $s(t)$ with $Q(s(t),t) < 0$ for all $t > 0$ we have $s < U_K$. Consider $s(t) = G_{\nu,\nu+2}(t)/t$, and write $\beta = \nu + 2$. Then $Q(s(t),t) < 0$ iff

$$\frac{1}{\left(\nu + \sqrt{t^2+\beta^2}\right)^2} + \frac{2(\nu+1)}{t^2}\frac{1}{\nu+\sqrt{t^2+\beta^2}} < \frac{1}{t^2} + \gamma,$$

which in turn is equivalent to

$$(1+\gamma t^2)\left(\nu+\sqrt{t^2+\beta^2}\right)^2 - 2(\nu+1)\left(\nu+\sqrt{t^2+\beta^2}\right) - t^2 > 0.$$

Let $\xi = \sqrt{t^2+\beta^2} - \beta$ so that $t \neq 0$ iff $\xi > 0$, $t^2 = (\xi+\beta)^2 - \beta^2 = \xi(\xi+2\beta)$, $\nu + \sqrt{t^2+\beta^2} = 2(\nu+1) + \xi$, and the inequality becomes

$$\begin{aligned} 0 < P(\xi) = {}& \gamma\xi^4 + \gamma(4(\nu+1)+2\beta)\xi^3 + (1 + 8(\nu+1)\beta\gamma + 4(\nu+1)^2\gamma - 1)\xi^2 \\ & + (4(\nu+1) + 8(\nu+1)^2\beta\gamma - 2(\nu+1) - 2\beta)\xi + (4(\nu+1)^2 - 4(\nu+1)^2). \end{aligned}$$

The coefficient of the linear term is 0, so that

$$P(\xi) = \gamma\xi^2(\xi^2 + (4(\nu+1)+2\beta)\xi + (8(\nu+1)\beta + 4(\nu+1)^2))$$

and for $\nu > -1$ we have $P(\xi) > 0$ for $\xi > 0$. Thus, $G_{\nu,\nu+2}(t)/t < U_K(t)$ for all $t > 0$. We thus have the following.

Theorem 2. *For all $t > 0$ and $\nu > -1$,*

$$\frac{G_{\nu,\nu+2}(t)}{t} < -\frac{\nu+1}{t^2} + \sqrt{\frac{(\nu+1)^2}{t^4} + \frac{1}{t^2} + \frac{1}{4(\nu+1)^2(\nu+2)}}.$$

Hence, the upper bound in Kokologiannaki [3, Theorem 2.1] is strictly weaker than the bound $f_\nu(t) = R_\nu(t)/t < G_{\nu,\nu+2}(t)/t$.

The various results can be summarized as follows: the "best" (in the sense of not being uniformly weaker than other) Amos-type bounds for R_ν currently available are

$$\begin{aligned} & G_{\nu+1/2,\nu+3/2} < R_\nu, \quad \nu \geq -1, \\ & R_\nu < G_{\nu,\nu+2}, \quad \nu \geq -1, \\ & R_\nu < G_{\nu+1/2,\sqrt{(\nu+1/2)(\nu+3/2)}}, \quad \nu > 0, \\ & R_\nu < G_{\nu+1/2,\nu+1/2}, \quad -1/2 \leq \nu \leq 0. \end{aligned}$$

4. Results

Theorem 3. *For $\nu \geq -1$,*

$$\mathcal{L}_\nu = \{(\alpha,\beta) : \alpha \geq \nu + 1/2, \alpha + \beta \geq 2(\nu+1), \beta \geq 0\}$$

and $G_{\nu+1/2,\nu+3/2}$ is the maximum of the family $\mathcal{G}_{\mathcal{L}_\nu}$ of lower Amos-type bounds for R_ν.

Proof. We already know that for $\nu \geq -1$, $G_{\nu+1/2,\nu+3/2} < R_\nu$. By Theorem 1, $G_{\alpha,\beta} \leq R_\nu$ is only possible if $\alpha + \beta \geq 2(\nu+1) = (\nu+1/2) + (\nu+3/2)$ and $\alpha \geq \nu + 1/2$. If $\beta < \nu + 3/2$, Lemma 3 implies that $G_{\alpha,\beta} < G_{\nu+1/2,\nu+3/2}$. Otherwise, we trivially have $G_{\alpha,\beta} \leq G_{\alpha,\nu+3/2} \leq G_{\nu+1/2,\nu+3/2}$. □

Theorem 4. *For $\nu \geq -1$, $\mathcal{U}_\nu$ is a closed convex set.*

Proof. For fixed $t > 0$, $(\alpha,\beta) \mapsto h_{\alpha,\beta}(t)$ is continuous, linear in α, and satisfies $\partial h_{\alpha,\beta}(t)/\partial\beta = \beta(t^2+\beta^2)^{-1/2} \geq 0$ and hence

$$\frac{\partial^2 h_{\alpha,\beta}(t)}{\partial\beta^2} = (t^2+\beta^2)^{-1/2} - \beta^2(t^2+\beta^2)^{-3/2} = t^2(t^2+\beta^2)^{-3/2} \geq 0$$

and is thus convex. By Theorem 1, $G_{\alpha,\beta} \geq R_\nu$ is only possible when $\alpha + \beta \geq 0$, for which it is equivalent to $h_{\alpha,\beta} \leq v_\nu$. Hence,

$$\mathcal{U}_\nu = \bigcap_{t>0}\{(\alpha,\beta) : h_{\alpha,\beta}(t) \leq v(t)\}$$

is the intersection of closed convex sets, and thus a closed convex set. □

Let

$$\mathcal{V}_\nu(\alpha) = \{\beta : (\alpha, \beta) \in \mathcal{U}_\nu\}$$
$$\beta^*_\nu(\alpha) = \sup \mathcal{V}_\nu(\alpha)$$
$$\alpha^*_\nu = \sup\{\alpha : \mathcal{V}_\nu(\alpha) \neq \emptyset\}.$$

As $\lim_{\beta\to\infty} G_{\alpha,\beta}(t) = 0$ for $t > 0$, clearly $\beta^*_\nu(\alpha) < \infty$ for $\nu \geq -1$.

Theorem 5. *For $\nu \geq -1$,*

$$\mathcal{U}_\nu = \{(\alpha, \beta) : \alpha \leq \alpha^*_\nu, \max(0, -\alpha) \leq \beta \leq \beta^*_\nu(\alpha)\},$$

with β^*_ν continuous, decreasing and concave.

Proof. For $\nu \geq -1$, we have $\beta \in \mathcal{V}_\nu(\alpha)$ iff $\alpha + \beta \geq 0$ and $h_{\alpha,\beta} \leq v_\nu$. Thus, as $h_{\alpha,\beta}$ is continuous and increasing in β, if $\mathcal{V}_\nu(\alpha)$ is non-empty, it is the closed interval $[\max(0, -\alpha), \beta^*_\nu(\alpha)]$. By Lemma 3, $G_{\alpha-\eta,\beta+\eta} > G_{\alpha,\beta}$ for all $\eta > 0$, so β^*_ν must be decreasing as long as $\mathcal{V}_\nu(\alpha)$ is non-empty. If $\alpha_n \uparrow \alpha^*_\nu$, $\beta_n = \beta^*_\nu(\alpha_n)$ is decreasing and non-negative and thus must have a finite limit β_∞. Taking limits in $\alpha_n + \beta_n \geq 0$ and $h_{\alpha_n,\beta_n} \leq v_\nu$ implies that $\alpha^*_\nu + \beta_\infty \geq 0$ and $h_{\alpha^*_\nu,\beta_\infty} \leq v_\nu$. Thus, $\mathcal{V}_\nu(\alpha^*_\nu)$ is non-empty. As $\mathcal{U}_\nu = \bigcup_\alpha \mathcal{V}_\nu(\alpha)$, the first assertion follows. Finally, as $\mathcal{U}_\nu$ is closed and convex, β^*_ν must be continuous and concave. □

Theorem 6. *Let $\nu \geq -1$. For $\alpha \leq \nu$, $\beta^*_\nu(\alpha) = 2(\nu + 1) - \alpha$. For $\nu < \alpha \leq \alpha^*_\nu$, $\beta^*_\nu(\alpha) < 2(\nu + 1) - \alpha$.*

Proof. We know that $(\nu, \nu + 2) \in \mathcal{U}_\nu$. By Theorem 1, $G_{\alpha,\beta} \geq R_\nu$ is only possible if $\alpha + \beta \leq 2(\nu + 1) = \nu + (\nu + 2)$ so that $\beta^*_\nu(\alpha) \leq 2(\nu + 1) - \alpha$. If $\alpha + \beta = 2(\nu + 1)$ and $\beta > 0$,

$$h_{\alpha,\beta}(t) = 2(\nu + 1) + \frac{t^2}{2\beta} + O(t^4), \quad t \to 0$$

by Eq. (3) and comparison with Eq. (2) shows that $h_{\alpha,\beta} \leq v_\nu$ is only possible if in fact $\beta \geq \nu + 2 > 0$, or equivalently, if $\alpha \leq 2(\nu + 1) - (\nu + 2) = \nu$. For $\alpha < \nu$, Lemma 3 implies that $G_{\alpha,2(\nu+1)-\alpha} > G_{\nu,\nu+2} \geq R_\nu$, so that indeed $\beta^*_\nu(\alpha) = 2(\nu + 1) - \alpha$. □

Let

$$Q_{\alpha,\beta}(s) = \beta^2 + (2(\nu + 1)\alpha - \alpha^2 - \beta^2)s + 2(\nu + 1/2 - \alpha)s^2.$$

Lemma 5. *Let $\Delta = v_\nu - h_{\alpha,\beta}$. Then*

$$t\Delta'(t) = \frac{Q_{\alpha,\beta}\left(\sqrt{t^2 + \beta^2}\right)}{\sqrt{t^2 + \beta^2}} + \big(2(\nu + 1) - v_\nu(t) - h_{\alpha,\beta}(t)\big)\Delta(t).$$

Proof. As shown in Simpson and Spector [9], v_ν satisfies the Riccati equation $tv'_\nu(t) = t^2 + 2(\nu + 1)v_\nu(t) - v_\nu(t)^2$ and clearly, $h'_{\alpha,\beta}(t) = t/\sqrt{t^2 + \beta^2}$. Hence, as $v^2 = h^2 + (v^2 - h^2) = h^2 + (v - h)(v + h)$,

$$\begin{aligned} tv'_\nu(t) &= t^2 + 2(\nu + 1)(\Delta(t) + h_{\alpha,\beta}(t)) - (h_{\alpha,\beta}(t)^2 + \Delta(t)(v_\nu(t) + h_{\alpha,\beta}(t))) \\ &= t^2 + 2(\nu + 1)h_{\alpha,\beta}(t) - h_{\alpha,\beta}(t)^2 + \big(2(\nu + 1) - v_\nu(t) - h_{\alpha,\beta}(t)\big)\Delta(t) \end{aligned}$$

with

$$\begin{aligned} t^2 + 2(\nu + 1)h_{\alpha,\beta}(t) - h_{\alpha,\beta}(t)^2 &= t^2 + 2(\nu + 1)\left(\alpha + \sqrt{t^2 + \beta^2}\right) - \left(\alpha^2 + 2\alpha\sqrt{t^2 + \beta^2} + t^2 + \beta^2\right) \\ &= 2(\nu + 1)\alpha - \alpha^2 - \beta^2 + 2(\nu + 1 - \alpha)\sqrt{t^2 + \beta^2} \end{aligned}$$

so that

$$\begin{aligned} t^2 + 2(\nu + 1)h_{\alpha,\beta}(t) - h_{\alpha,\beta}(t)^2 - \frac{t^2}{\sqrt{t^2 + \beta^2}} &= \frac{(2(\nu + 1)\alpha - \alpha^2 - \beta^2)\sqrt{t^2 + \beta^2} + 2(\nu + 1 - \alpha)(t^2 + \beta^2) - t^2}{\sqrt{t^2 + \beta^2}} \\ &= \frac{Q_{\alpha,\beta}\left(\sqrt{t^2 + \beta^2}\right)}{\sqrt{t^2 + \beta^2}}, \end{aligned}$$

whence the lemma. □

Let

$$\alpha_\nu^\flat = \min(\nu + 1/2, 2\nu + 1)$$

(so that $\alpha_\nu^\flat$ equals $\nu + 1/2$ for $\nu \geq -1/2$ and $2\nu + 1$ otherwise), and for $-1 \leq \nu \leq \alpha \leq \alpha_\nu^\flat$ let

$$\beta_\nu^\flat(\alpha) = \sqrt{2\nu + 1 - 2\alpha} + \sqrt{2\nu + 1 + 2\nu\alpha - \alpha^2} = \sqrt{2(\nu + 1/2 - \alpha)} + \sqrt{(\alpha + 1)(2\nu + 1 - \alpha)}$$

(where the second expressions shows that $\beta_\nu^\flat$ is well-defined).

Lemma 6. *Let $\nu \geq -1$. Then $\beta_\nu^\flat$ is strictly concave with $\beta_\nu^\flat(\nu) = \nu + 2$, $\beta_\nu^\flat(\alpha_\nu^\flat)$ equals $\sqrt{(\nu + 1/2)(\nu + 3/2)}$ if $\nu \geq -1/2$ and $\sqrt{-2(\nu + 1/2)}$ if $-1 \leq \nu \leq -1/2$, and $\alpha \mapsto \alpha + \beta_\nu^\flat(\alpha)$ is non-negative and decreasing.*

Proof. The assertions about the values of $\beta_\nu^\flat$ at ν and $\alpha_\nu^\flat$ are straightforward. If $\nu = -1$, $\alpha_\nu^\flat = \nu$ and there is nothing left to prove. Hence, take $\nu > -1$. The second derivative of $\alpha \mapsto \sqrt{f(\alpha)}$ is given by

$$\frac{d^2\sqrt{f(\alpha)}}{d\alpha^2} = \frac{f''(\alpha)f(\alpha) - f'(\alpha)^2/2}{2\sqrt{f(\alpha)}^3}.$$

For $f_1(\alpha) = 2(\nu + 1/2 - \alpha)$ and $f_2(\alpha) = 2\nu + 1 + 2\nu\alpha - \alpha^2$ we have $f_1'(\alpha) = -2$, $f_1''(\alpha) = 0$, $f_2'(\alpha) = 2(\nu - \alpha)$ and $f_2''(\alpha) = -2$, giving numerators -2 and $-2(2\nu + 1 + 2\nu\alpha - \alpha^2) - 4(\nu - \alpha)^2/2 = -2(\nu + 1)^2 < 0$. Hence $\beta_\nu^\flat$ is the sum of two strictly concave functions, and thus strictly concave. Clearly,

$$\frac{d\beta_\nu^\flat(\alpha)}{d\alpha} = \frac{-1}{\sqrt{2\nu + 1 - 2\alpha}} + \frac{\nu - \alpha}{\sqrt{2\nu + 1 + 2\nu\alpha - \alpha^2}}$$

with value -1 at $\alpha = \nu$. By strict concavity, the derivative of $\beta_\nu^\flat$ is decreasing, and hence less than -1 for $\alpha > \nu$, so that the derivative of $\alpha \mapsto f(\alpha) = \alpha + \beta_\nu^\flat(\alpha)$ is negative for $\alpha > \nu$ and f is decreasing. It remains to show that $f(\alpha_\nu^\flat) \geq 0$. If $\nu \geq -1/2$, this is immediate from $\alpha_\nu^\flat = \nu + 1/2 \geq 0$. Otherwise, $\alpha_\nu^\flat = 2\nu + 1 < 0$ and $f(\alpha_\nu^\flat) = 2\nu + 1 + \sqrt{-(2\nu + 1)}$, which is non-negative as $0 \leq -(2\nu + 1) \leq 1$. □

Theorem 7. *Let $\nu \geq -1$. Then for $\nu \leq \alpha \leq \alpha_\nu^\flat$, $G_{\alpha,\beta_\nu^\flat(\alpha)} \geq R_\nu$.*

Proof. The proof will be based on the ideas of Simpson and Spector [9]. Suppose Δ is sufficiently often continuously differentiable on $[0, \infty)$ with $\Delta(0) > 0$. Suppose that for all $t > 0$, $\Delta(t) = 0$ implies that there exists a suitable odd k such that $\Delta^{(l)}(t) = 0$ for $l < k$ and $\Delta^{(k)}(t) > 0$. Then $\Delta(t) \geq 0$ for all $t \geq 0$, as otherwise for $s = \inf\{t > 0 : \Delta(t) = 0\}$ we would have $\Delta(s - \epsilon) = \Delta^{(k)}(s^*)(-\epsilon)^k/k! < 0$ for all sufficiently small $\epsilon > 0$ and a suitable $s^* \in (s - \epsilon, s)$, which is impossible.

In our case, $\Delta = v_\nu - h_{\alpha,\beta}$, where $\beta = \beta_\nu^\flat(\alpha)$. If $\alpha = \nu$, we have $\beta = \nu + 2$ and we already know for $\nu \geq -1$ that $G_{\alpha,\beta} = G_{\nu,\nu+2} \geq R_\nu$. By Lemma 6, $\alpha + \beta_\nu^\flat(\alpha)$ is decreasing and hence maximal for $\alpha = \nu$ with value $2(\nu + 1)$. Thus, for $\alpha > \nu$ we have $\alpha + \beta_\nu^\flat(\alpha) < 2(\nu + 1)$, or equivalently, $\Delta(0) > 0$.

Write $s(t) = \sqrt{t^2 + \beta^2}$. If $\alpha = \nu + 1/2$, which is only possible if $\nu \geq -1/2$, we have $\beta = \sqrt{(\nu + 1/2)(\nu + 3/2)}$ and $Q_{\alpha,\beta} = \beta^2$ for all s. If $\nu = -1/2$, we already know that $R_{-1/2} = \tanh \leq G_{0,0}$. Otherwise, $Q_{\alpha,\beta}(s) = \beta^2 > 0$. If $\Delta(t) = 0$ for some $t > 0$, Lemma 5 implies that $\Delta'(t) = \beta^2/(ts(t)) > 0$, completing the proof for this case.

Hence, consider the case where $\nu < \alpha < \nu + 1/2$. Solving $Q_{\alpha,\beta}(s) = 0$ has discriminant

$$\begin{aligned}&(2(\nu + 1)\alpha - \alpha^2 - \beta^2)^2 - 8(\nu + 1/2 - \alpha)\beta^2 \\ &\quad= \left(2(\nu + 1)\alpha - \alpha^2 - \beta^2 + 2\beta\sqrt{2\nu + 1 - 2\alpha}\right) \cdot \left(2(\nu + 1)\alpha - \alpha^2 - \beta^2 - 2\beta\sqrt{2\nu + 1 - 2\alpha}\right),\end{aligned}$$

with $\beta = \beta_\nu^\flat(\alpha)$ the larger root of the first factor. Hence, the discriminant vanishes, and with

$$\sigma = -\frac{2(\nu + 1)\alpha - \alpha^2 - \beta^2}{4(\nu + 1/2 - \alpha)} = \frac{2\sqrt{2\nu + 1 - 2\alpha}\beta}{4(\nu + 1/2 - \alpha)} = \frac{\beta}{\sqrt{2\nu + 1 - 2\alpha}} > 0$$

we have $Q_{\alpha,\beta}(s) = \gamma(s - \sigma)^2$, where $\gamma = 2\nu + 1 - 2\alpha > 0$.

If $\Delta(t) = 0$ for some $t > 0$, Lemma 5 implies that $t\Delta'(t) = Q_{\alpha,\beta}(s(t))/s(t)$. If $s(t) \neq \sigma$, $Q_{\alpha,\beta}(s(t)) > 0$, and the proof is complete. Otherwise, use Lemma 5 to write $t\Delta'(t) = \xi(t) + \eta(t)\Delta(t)$, where

$$\xi(t) = \gamma(s(t) - \sigma)^2/s(t) = \gamma\left(s(t) - 2\sigma + \frac{\sigma^2}{s(t)}\right)$$

so that $\xi'(t) = \gamma(s'(t) - \sigma^2 s'(t)/s(t)^2)$ and

$$\xi''(t) = \gamma\left(s''(t) - \sigma^2\left(\frac{s''(t)}{s(t)^2} - 2\frac{s'(t)^2}{s(t)^3}\right)\right).$$

If $s(t) = \sigma$, $\xi'(t) = 0$ and $\xi''(t) = 2\gamma s'(t)^2/\sigma > 0$. Differentiation gives $\Delta'(t) + t\Delta''(t) = \xi'(t) + \eta'(t)\Delta(t) + \eta(t)\Delta'(t)$ and $2\Delta''(t) + t\Delta'''(t) = \xi''(t) + \eta''(t)\Delta(t) + 2\eta'(t)\Delta'(t) + \eta(t)\Delta''(t)$, so that if $s(t) = \sigma$, $\Delta(t) = \Delta'(t) = \Delta''(t) = 0$ and $\Delta'''(t) = \xi''(t)/t > 0$, and the proof is complete. □

Theorem 8. *Let $\nu \geq -1$. Then the elements of $\{G_{\alpha,\beta_\nu^\flat(\alpha)} : \nu \leq \alpha \leq \alpha_\nu^\flat\}$ are mutually incomparable.*

Proof. By Lemma 6, $\alpha \mapsto \alpha + \beta_\nu^\flat(\alpha)$ is decreasing, whence the result by using Lemma 3. □

Theorem 9. *For $\nu \geq -1/2$, $\alpha_\nu^* = \nu + 1/2$ and*

$$\beta_\nu^*(\nu + 1/2) = \beta_\nu^\flat(\nu + 1/2) = \sqrt{(\nu + 1/2)(\nu + 3/2)}.$$

For $-1 \leq \nu < -1/2$, $\alpha_\nu^* < \nu + 1/2$.

Proof. Let $\beta^\flat = \beta_\nu^\flat(\nu + 1/2) = \sqrt{(\nu + 1/2)(\nu + 3/2)}$. For arbitrary β,

$$G_{\nu+1/2,\beta} = 1 - \frac{\nu + 1/2}{t} + \frac{2(\nu + 1/2)^2 - \beta^2}{2t^2} + O(t^{-3}), \quad t \to \infty$$

by Eq. (4) and comparison with Eq. (1) shows that $G_{\nu+1/2,\beta} \geq R_\nu$ is only possible if

$$2(\nu + 1/2)^2 - \beta^2 \geq (\nu + 1/2)(\nu - 1/2),$$

or equivalently, if $\beta^2 \leq 2(\nu+1/2)^2 - (\nu+1/2)(\nu-1/2) = (\nu+1/2)(\nu+3/2)$. For $\nu < -1/2$, the upper bound is negative, so that $G_{\nu+1/2,\beta} \geq R_\nu$ is impossible for all $\beta \geq 0$ and hence $\alpha_\nu^* < \nu + 1/2$. For $\nu \geq 1/2$, the condition is equivalent to $\beta \leq \beta^\flat$. By Theorem 7, $(\nu + 1/2, \beta^\flat) \in \mathcal{U}_\nu$ and by Theorem 1, $\alpha \leq \nu + 1/2$, so that $\alpha_\nu^* = \nu + 1/2$ and $\beta_\nu^*(\nu + 1/2) = \beta^\flat$. □

Theorem 10. *Let $\nu \geq -1/2$ and $\nu < \alpha < \nu + 1/2$. Then there exists a unique positive $t_\nu^*(\alpha)$ at which $G_{\alpha,\beta_\nu^*(\alpha)}$ is tangent to R_ν. The map $\alpha \mapsto t_\nu^*(\alpha)$ is continuous and increasing on $(\nu, \nu + 1/2)$, with $\lim_{\alpha\to\nu+} t_\nu^*(\alpha) = 0$ and $\lim_{\alpha\to\nu+1/2-} t_\nu^*(\alpha) = \infty$.*

Proof. Write $\beta^* = \beta_\nu^*(\alpha)$. By Theorem 6, we can find $\delta > 0$ such that $\beta^* \leq 2(\nu + 1) - \alpha - \delta$. Using Lemma 3 and the fact that $\sqrt{(\nu + 1/2)(\nu + 3/2)} \leq \beta_\nu^\flat(\alpha) \leq \beta^*$, we can find $0 < t_1 < t_2$ such that for all $\beta^* \leq \beta \leq \beta^* + \delta$, $G_{\alpha,\beta}(t) \geq G_{\nu,\nu+2}(t) > R_\nu(t)$ for $0 < t \leq t_1$ and $G_{\alpha,\beta}(t) \geq G_{\nu+1/2,\sqrt{(\nu+1/2)(\nu+3/2)}}(t) > R_\nu(t)$ for $t \geq t_2$. If $G_{\alpha,\beta^*} > R_\nu$, we have for all $\eta > 0$ sufficiently small that $G_{\alpha,\beta^*+\eta}(t) \geq R_\nu(t)$ for $t_1 \leq t \leq t_2$. By the above, the same holds true for $0 < t \leq t_1$ and $t \geq t_2$. Hence, $G_{\alpha,\beta^*+\eta} \geq R_\nu$ for all $\eta > 0$ sufficiently small, which contradicts the maximality of β^*. Thus, there must be at least one $t > 0$ such that $G_{\alpha,\beta^*}(t) = R_\nu(t)$, and clearly, the derivatives must agree at t as otherwise G_{α,β^*} could not be an upper bound for R_ν. Equivalently, $h_{\alpha,\beta}$ must be tangent to v_ν at t. By Lemma 5, this is the case iff t solves $Q_{\alpha,\beta^*}\left(\sqrt{t^2 + \beta^{*2}}\right) = 0$, from which we infer that $t = t_\nu^*(\alpha)$ is uniquely determined and continuous as a function of α. The limits for $\alpha \to \nu$ from the right and $\alpha \to \nu + 1/2$ from the left are obvious. To show that t^* is increasing, it suffices to show that it is injective. Hence, let $\nu < \alpha_1 < \alpha_2 < \nu + 1/2$ and suppose that $t_\nu^*(\alpha_1) = t_\nu^*(\alpha_2) = t^*$. Then with $\beta_i^* = \beta_\nu^*(\alpha_i)$, the h_{α_i,β_i^*} must have the same value and derivative at t^*, so that

$$\frac{t^*}{\sqrt{t^{*2} + \beta_1^{*2}}} = \frac{t^*}{\sqrt{t^{*2} + \beta_2^{*2}}},$$

and hence $\beta_1^* = \beta_2^*$, which is impossible as β_ν^* is decreasing by Theorem 5. □

Theorem 11. *Let $\nu \geq -1/2$. Then $\{G_{\alpha,\beta_\nu^*(\alpha)} : \nu \leq \alpha \leq \nu + 1/2\}$ are the minimal elements of the family $\mathcal{G}_{\mathcal{U}_\nu}$ of upper Amos-type bounds for R_ν, and*

$$R_\nu = \min\{G_{\alpha,\beta_\nu^*(\alpha)} : \nu \leq \alpha \leq \nu + 1/2\}.$$

Proof. Let $t > 0$. By Theorem 10, there exists a unique $\nu < \alpha < \nu + 1/2$ so that $t_\nu^*(\alpha) = t$ and hence $R_\nu(t) = G_{\alpha,\beta_\nu^*(\alpha)}(t)$, proving the second assertion. Let $\nu \leq \alpha_1 < \alpha_2 \leq \nu + 1/2$ and $\beta_i^* = \beta_\nu^*(\alpha_i)$. If $\alpha_1 = \nu$, Theorem 6 shows that $2(\nu + 1) = \alpha_1 + \beta_1^* > \alpha_2 + \beta_2^*$. If $\alpha_1 > \nu$ and $\alpha_1 + \beta_1^* \leq \alpha_2 + \beta_2^*$, Lemma 3 implies that $R_\nu \leq G_{\alpha_2,\beta_2^*} < G_{\alpha_1,\beta_1^*}$, which is impossible as by Theorem 10, G_{α_1,β_1^*} must be the only tangent to R_ν at $t_\nu^*(\alpha_1)$. Thus we always have $\alpha_1 + \beta_1^* > \alpha_2 + \beta_2^*$, and again by Lemma 3, there always exists $t = t(\alpha_1, \alpha_2)$ such that $G_{\alpha_1,\beta_1^*}(s) < G_{\alpha_2,\beta_2^*}(s)$ for $0 < s < t$ and $G_{\alpha_1,\beta_1^*}(s) > G_{\alpha_2,\beta_2^*}(s)$ for $s > t$. As $G_{\alpha,\beta_\nu^*(\alpha)} > G_{\nu,\nu+2} = G_{\nu,\beta_\nu^*(\nu)}$ for $\alpha < \nu$ and trivially $G_{\alpha,\beta} \geq G_{\alpha,\beta^*(\alpha)}$ provided that $(\alpha, \beta) \in \mathcal{U}_\nu$, the first assertion follows, and the proof is complete. □

Finally, let us consider the cases where $\nu = -k$ is a negative integer. As readily seen from the series expansion, $I_{-k} = I_k$, and hence $R_{-k} = I_{-k+1}/I_{-k} = I_{k-1}/I_k = 1/R_{k-1}$.

Theorem 12. *If k is a positive integer,*

$$\mathcal{U}_{-k} = \{(-\beta, \beta) : \beta \geq k\}$$

and $G_{-k,k}$ is the minimum of the family $\mathcal{G}_{\mathcal{U}_\nu}$ of upper Amos-type bounds for R_{-k}.

Proof. As $R_{-k} > 0$ and has a pole at $t = 0$, the same must be true for upper bounds $G_{\alpha,\beta}$ of R_{-k}, implying that necessarily $\alpha + \beta = 0$. As

$$G_{-\beta,\beta}(t) = \frac{t}{\sqrt{t^2+\beta^2}-\beta} = \frac{t\left(\sqrt{t^2+\beta^2}+\beta\right)}{(t^2+\beta^2)-\beta^2} = \frac{\sqrt{t^2+\beta^2}+\beta}{t} = \frac{1}{G_{\beta,\beta}(t)},$$

we have $1/G_{\beta,\beta} = G_{-\beta,\beta} \geq R_{-k} = 1/R_{k-1}$ iff $R_{k-1} \geq G_{\beta,\beta}$, i.e., $(\beta, \beta) \in \mathcal{L}_{k-1}$. From the characterization of $\mathcal{L}_\nu$ for $\nu \geq -1$ (Theorem 3), this is possible iff $\beta \geq k - 1/2$ and $2\beta \geq 2k$, or equivalently, $\beta \geq k$. □

Theorem 13. *If k is a positive integer,*

$$\mathcal{L}_{-k} = \{(\alpha, \beta) : \alpha \geq -(k - 1/2), \alpha + \beta \geq 0, \beta \geq 0\}$$

and $G_{-(k-1/2),k-1/2}$ is the maximum of the family $\mathcal{G}_{\mathcal{L}_\nu}$ of lower Amos-type bounds for R_{-k}.

Proof. For lower bounds $G_{\alpha,\beta}$ of R_{-k}, we must have $\alpha + \beta \geq 0$ by the usual arguments, and Theorem 1 implies that necessarily $\alpha \geq -k + 1/2$. On the other hand, we also know that $R_{k-1} \leq G_{k-1/2,k-1/2}$, or equivalently, $G_{-(k-1/2),k-1/2} \leq R_{-k}$, and the proof is complete. □

Note that for $k = 1$, we already know by Theorem 3 that $G_{-1+1/2,-1+3/2} = G_{-1/2,1/2}$ is the greatest lower bound for R_{-1}, and Theorem 6 yields that $\beta^*_{-1}(-1) = 1$, so that $G_{-1,1}$ is the least upper bound for R_{-1} with $\alpha = -1$.

5. Summary and conclusions

In this paper, we systematically investigate lower and upper Amos-type bounds for $R_\nu = I_{\nu+1}/I_\nu$ on the positive reals when R_ν is positive, or equivalently, when $\nu \geq -1$ or ν is a negative integer.

For $\nu \geq -1$, the set $\mathcal{L}_\nu$ of all (α, β) giving lower bounds $G_{\alpha,\beta} \leq R_\nu$ has a simple explicit description, and $G_{\nu+1/2,\nu+3/2}$ is the maximum of the family $\mathcal{G}_{\mathcal{L}_\nu}$ of lower Amos-type bounds for R_ν (Theorem 3).

For $\nu \geq -1$, the set $\mathcal{U}_\nu$ of all (α, β) giving upper bounds $G_{\alpha,\beta} \geq R_\nu$ is of the form $\{(\alpha, \beta) : \alpha \leq \alpha^*_\nu, \max(0, -\alpha) \leq \beta \leq \beta^*_\nu(\alpha)\}$, where $\nu \leq \alpha^*_\nu \leq \nu + 1/2$ and β^*_ν is continuous, decreasing and concave (Theorem 5), with $\beta^*_\nu(\nu) = \nu + 2$ and $\alpha + \beta^*_\nu(\alpha) < 2(\nu + 1)$ for $\alpha > \nu$ (Theorem 6). If $\nu \geq -1/2$, $\alpha^*_\nu = \nu + 1/2$ and $\beta^*_\nu(\nu + 1/2) = \sqrt{(\nu + 1/2)(\nu + 3/2)}$ by Theorem 9, and the upper bounds in the family $\{G_{\alpha,\beta^*_\nu}(\alpha), \nu \leq \alpha \leq \nu + 1/2\}$ are tangent to R_ν in exactly one point $t^*_\nu(\alpha)$ (Theorem 10, taking $t^*_\nu(\nu) = 0$ and $t^*_\nu(\nu + 1/2) = \infty$), and the minimal elements of the family $\mathcal{G}_{\mathcal{U}_\nu}$ of upper Amos-type bounds for R_ν, with R_ν as their lower envelope (Theorem 11).

Thus, for $\nu \geq -1$, the pointwise maximum over all lower Amos-type bounds equals $G_{\nu+1/2,\nu+3/2} < R_\nu$, and hence is always smaller than R_ν. On the other hand, for $\nu \geq -1/2$, the pointwise minimum over all upper Amos-type bounds equals R_ν.

For $\nu \geq -1$ and $\nu \leq \alpha < \alpha^\flat_\nu = \min(\nu + 1/2, 2\nu + 1)$, Theorems 7 and 8 establish a family $\{G_{\alpha,\beta^\flat_\nu}(\alpha), \nu \leq \alpha \leq \alpha^\flat_\nu\}$ of explicitly computable, mutually incomparable upper bounds for R_ν with $\beta^\flat_\nu(\nu) = \beta^*_\nu(\nu) = \nu + 2$. For $\nu < \alpha < \alpha^\flat_\nu$, these bounds are new. For $\nu \geq -1/2$, $\alpha^\flat_\nu = \alpha^*_\nu = \nu + 1/2$ and $\beta^\flat_\nu(\nu + 1/2) = \beta^*_\nu(\nu + 1/2)$, and Theorem 7 extends the range of the bound $G_{\nu+1/2,\sqrt{(\nu+1/2)(\nu+3/2)}} \geq R_\nu$ given in Simpson and Spector [9] from $\nu \geq 0$ to $\nu \geq -1/2$, and for $-1/2 < \nu < 0$ dominates $G_{\nu+1/2,\nu+1/2}$ as the best previously available upper bound with $\alpha = \nu + 1/2$ (and hence first order exact as $t \to \infty$).

Finally, for the cases where $\nu = -k$ is a negative integer, Theorems 12 and 13 give explicit characterizations of $\mathcal{U}_{-k}$ and $\mathcal{L}_{-k}$, and establish $G_{-k,k}$ and $G_{-(k-1/2),k-1/2}$ as the least upper and greatest lower Amos-type bounds for R_{-k}, respectively.

For $-1 \leq \nu < -1/2$, the value of α^*_ν is not known; the results in this paper imply that $\alpha^\flat_\nu \leq \alpha^*_\nu < \nu + 1/2$. It is also not known whether in this case R_ν can be obtained as the lower envelope of all upper Amos-type bounds. For $\nu = -1$, this is certainly not the case (as $G_{-1,1}$ is the uniformly smallest upper bound). Hence the range $-1 < \nu < -1/2$ deserves further investigation.

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