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A Two Period Model with Portfolio Choice: Understanding Results from Different Solution Methods

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Abstract

Using a stylized two period model we obtain portfolio solutions from two solution approaches that belong to the class of local approximation methods – the approach of Judd and Guu (2001, hereafter 'JG') and the approach of Devereux and Sutherland (2010, 2011, hereafter 'DS') – and compare them with the true portfolio solution. We parameterize the model to match mean, standard deviation, skewness and kurtosis of return data on aggregate MSCI stock market indices. The optimal equity holdings in the true solution depend on the size of uncertainty, and the precise form of this relationship is determined by the distributional properties of equity returns.

While the DS method and the JG approach provide the same portfolio solution as the size of uncertainty goes to zero, else the two solutions can differ substantially. Because under the DS method portfolio holdings are never approximated in the direction of the size of uncertainty, even higher-order approximations lead to the (zero-order) constant solution in our example model. In contrast, the JG solution generally varies as the size of uncertainty changes, and already a second-order JG solution can account for effects of skewness and kurtosis of equity returns.

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1 Introduction

In this paper we contrast the performance of two approaches to computing portfolios that belong to the class of local approximation methods – the bifurcation approach suggested by Judd and Guu (2001, hereafter 'JG') and the solution method of Devereux and Sutherland (2010, 2011, hereafer 'DS') – with the true portfolio solution in a stylized two period model. The DS solution approach has received considerable attention in solving portfolio problems in dynamic macroeconomic models in the recent past. While its main advantages lie in obtaining portfolio solutions in a dynamic setting, the two period setting of the present paper is already able to shed light on some of its properties.

We present a model, which closely follows Judd and Guu (2001), where there are two countries (agents) which can hold a risky asset (equity) or a safe asset (bond), and country 1 is less risk averse than country 2. In this setting, country 1 chooses to take on more of the risky asset and goes short in the safe asset. We show that, in the true solution, optimal equity holdings generally depend on the size of uncertainty, and that the precise way in which they do so depend on the distributional properties (skewness, kurtosis, and higher order moments) of equity returns. We show how each solution method performs as we vary the size of uncertainty, and when the model is realistically calibrated to moments of return data of various MSCI aggregate stock market indices.

Our results are as follows: Because the DS' portfolio solution is never approximated in the direction of the size of uncertainty, applying the DS solution method (up to any order) always delivers the constant (zero-order) portfolio solution in our two-period model. The JG bifurcation method, on the other hand, performs substantially better in this respect: we show that the zero-order portfolio solution coincides with DS; but higher-order JG solutions do account for variations of the size of uncertainty, and already a second-order JG solution is able to account for the effects of skewness or kurtosis of equity returns on the solution.

The maximum differences with respect to the true solution occur in a calibration to stock market return moments of the MSCI Pacific ex-Japan: in this case the differences amount to -2.31% for the second-order JG solution, and to -6.13% for the DS solution, in our baseline setting where country 2 is twice as risk averse as country 1. In some sensitivity analysis where country 2 is three times as risk averse as country 1, those differences rise to -3.14% for the (second-order) JG solution, and to -7.56% for the DS solution.

2 Model

The world consists of two countries with a representative investor in each and a single consumption good. Each investor lives for 2 periods. There is no consumption in period 1, only portfolio decisions: agents trade assets in period 1 and consume the asset payoffs in period 2. There are two assets available for trade: equity – a claim on the total world's output –, and a risk-free bond. The bond yields one unit of consumption in period 2 and serves as a numeraire, i.e., the period 1 bond price is normalized to 1. Each share has price p in period 1 and has a random period 2 value, $Y = 1 + \varepsilon z$. We assume $E\{z\} = 0$ and $E\{z^2\} = 1$. In addition, we assume that the support for z is bounded from below, so that $Y > 0$ for all values of ε and z.

Each investor i starts with b_i^0 units of bonds and θ_i^0 shares of equity. Investors' utility is assumed to be of the constant relative risk aversion type, $u_i(C_i) = C_i^{1-\gamma_i}/(1-\gamma_i)$. C_i

denotes investor i's consumption in period 2 which equals her final wealth. Without loss of generality, we assume $\theta_1^0 + \theta_2^0 = 1$; this implies that z denotes aggregate risk in the world endowment Y. Let θ_i be the shares of equity and b_i the value of bonds held by trader i after trading in period 1. Each investor i solves:

$$\max_{\theta_i, b_i} Eu_i(C_i)$$

s.t.:

$$\theta_i^0 p + b_i^0 = \theta_i p + b_i \quad \text{(budget constraint in period 1)}$$

$$C_i = \theta_i Y + b_i, \forall Y \quad \text{(budget constraints in period 2)}$$

Market-clearing implies $\theta_1 + \theta_2 = 1$, $b_1 + b_2 = 0$. Define $\theta = \theta_1$; then $\theta_2 = 1 - \theta$. Also, denote $b_1 = b = -b_2$. We do similarly for initial endowments, i.e. $\theta^0 = \theta_1^0$, $\theta_2^0 = 1 - \theta^0$, and $b_1^0 = b^0 = -b_2^0$.

Equilibrium is characterized by the following system of equations:

$$\begin{array}{ll}
\text{(E1)}: \lambda_1 = E\left[u_1'(C_1)\right], & \text{(E2)}: \lambda_2 = E\left[u_2'(C_2)\right], \\
\text{(E3)}: p\lambda_1 = E\left[u_1'(C_1)Y\right], & \text{(E4)}: p\lambda_2 = E\left[u_2'(C_2)Y\right], \\
\text{(E5)}: C_1 = \theta Y + b, \forall Y, & \text{(E6)}: C_2 = (1-\theta)Y - b, \forall Y, \\
\text{(E7)}: \theta^0 p + b^0 = \theta p + b, &
\end{array}$$

with unknowns: $C_1, C_2, \theta, b, p, \lambda_1, \lambda_2$; λ_i denotes the Lagrange multiplier on investor i's period 1 budget constraint. In addition, denote the return on equity by $R_e = Y/p$, bond return $R_b = 1$, and excess return, $R_x = R_e - R_b$. The above equilibrium conditions can be further reduced to a system of two equations in variables θ and p, which we define as:

$$H\left(\theta\left(\varepsilon\right), p\left(\varepsilon\right), \varepsilon\right) = \left[\begin{array}{c} E\left[u_1'(\theta Y + b^0 + (\theta^0 - \theta)p)(Y - p)\right] \\ E\left[u_2'((1-\theta)Y - b^0 - (\theta^0 - \theta)p)(Y - p)\right] \end{array}\right] = \mathbf{0}. \tag{1}$$

2.1 Nonlinear portfolio solution

To obtain the nonlinear (quadrature) portfolio solution in this simple economy, called 'true solution' hereafter, we approximate the expectations operator using quadrature methods and simply solve the system given in (1) using a nonlinear equations solver.[1] Appendix A provides further details.

2.2 Devereux-Sutherland portfolio solution

The contributions by Devereux and Sutherland (2011, 2010) provide easy-to-apply methods to obtain approximate portfolio solutions in a *dynamic* stochastic GE model. While we apply their method in a model that is essentially static in the sense that there is no variation in state variables, it is indicative to reflect first on how their method works in the general case of a dynamic setting. In particular, denote with α_t the true (unknown) function of optimal

[1] The nonlinear solution in this static economy is simple to obtain. In more general, dynamic settings nonlinear methods providing a globally valid approximation for portfolios is substantially more complex. Such global portfolio solution methods have been proposed by Kubler and Schmedders (2003).

holdings of any asset that is zero-net supply.[2] In the above contributions, DS show that a zero-order (first-order) approximation to the true portfolio solution can be obtained from a second (third) order Taylor series expansion to the model's portfolio optimality conditions, in conjunction with a first (second) order Taylor series expansion to the model's other optimality and equilibrium conditions. Applying these steps one obtains an approximate portfolio solution of the format:

$$\alpha_t = \bar{\alpha} + \alpha' \widehat{x}_t. \tag{2}$$

where $\bar{\alpha}$ is the zero-order (constant) part of the solution, α' is a vector of the first-order coefficients, x_t is the vector of the model's state variables, and $\widehat{x}_t$ refers to the state variables expressed as (log-)deviations from their steady state values.

DS also state that their solution principle, which builds up on earlier work by Samuelson (1970), could be successively applied to higher orders: to obtain an n-th order accurate portfolio solution, one needs to approximate the portfolio optimality conditions up to order $n+2$, in conjunction with an approximation to the model's other optimality and equilibrium conditions of order $n+1$. E.g., going one order higher, one would obtain the approximate portfolio solution as $\alpha_t = \bar{\alpha} + \alpha' \widehat{x}_t + \frac{1}{2} \widehat{x}_t' \alpha'' \widehat{x}_t$.

It is important to realize that the expression in equation (2) is, however, not the same as what would result from a Taylor series expansion of the true policy function α_t. Following Schmitt-Grohé and Uribe (2004) and Jin and Judd (2002) we can think of the true policy function in a recursive economy as a function that depends on the model's state variables, x_t, and on a parameter that scales the variance-covariance matrix of the model's exogenous shock processes, ε; that is, $\alpha_t = \alpha(x_t, \varepsilon)$. A Taylor series to policy function α_t, evaluated at approximation points $x_t = \overline{x}$ and $\varepsilon = 0$, would then result in:

$$\alpha_t = \alpha(\overline{x}, 0) + \alpha_x(\overline{x}, 0)\widehat{x}_t + \alpha_\varepsilon(\overline{x}, 0)\varepsilon + \frac{1}{2}\widehat{x}_t' \alpha_{xx}(\overline{x}, 0)\widehat{x}_t + \alpha_{x\varepsilon}(\overline{x}, 0)\widehat{x}_t \varepsilon + \frac{1}{2}\alpha_{\varepsilon\varepsilon}(\overline{x}, 0)\varepsilon^2 + ... \tag{3}$$

That is, in contrast to the Taylor series expansion in equation (3) the DS approximate portfolio solution does only consider how variations in the model's state variables affect the optimal portfolio solution, but ignores the effect of variations in the size of uncertainty.[3,4]

Let us return to finding the DS portfolio solution in our two period model. To apply their method, it is convenient to reformulate the portfolio positions in zero-sum value terms. In our model, this means defining portfolio positions as:

$$\alpha_e = (\theta - \theta^0)p, \quad \alpha_b = b - b^0.$$

[2] DS' exposition of their method is in terms of assets in zero-net supply. This is not in any way restrictive. For assets in positive net supply, such as equities, this can be easily achieved by defining portfolio positions in terms of deviations from some initial portfolio endowments, and then multiplying them by their price.

[3] The comparison of the DS solution with equation (3) is simply for reasons of exposition. We are of course not suggesting that an approximate solution to the true unknown portfolio function actually can be obtained by taking a simple Taylor series expansion around the non-stochastic steady state. This is not feasible using standard local approximation methods (using the standard implicit function theorem) – the portfolio is indeterminate both at the non-stochastic steady state and in a first-order approximation of the stochastic setting. This is exactly the problem that the DS method and the JG method have addressed and proposed (different) ways of solving for.

[4] In the general case of a *dynamic* model, this still does not imply that the size of uncertainty cannot have an effect on optimal portfolios. In principle there could be an effect of the size of uncertainty, ε, on the portfolio through the effect of ε on the states themselves. This, however, would only be happening at higher orders, as the (state) variables are not affected by ε at first-order (certainty equivalence) and only through a constant at second-order (see Schmitt-Grohé and Uribe (2004)).

We obtain the zero-order or constant portfolio solution, $\bar{\alpha}_e$, from the second-order approximation of both countries' first order optimality conditions with respect to portfolio allocations[5], which, once combined, result in an expression that contains only first-order terms of the model's macro variables. This equation takes the following form:

$$E\left[\left(\gamma_1 \hat{c}_1 - \gamma_2 \hat{c}_2\right) \hat{r}_x\right] = 0. \tag{4}$$

The application of the DS method in our case is greatly simplified – we do not need to obtain the full dynamic solution to the macro variables of the model. After (log-)linearizing country 1's budget constraint, the economy-wide resource constraint and the definition of excess returns, we can directly express the relevant macro variables, $\hat{c}_1$, $\hat{c}_2$ and $\hat{r}_x$, in terms of the exogenous shock, $\hat{y}$. This allows us to derive the following expression for the zero-order portfolio, $\bar{\alpha}_e$ (details are provided in appendix B):

$$\overline{\alpha}_e = \frac{\gamma_2 - \gamma_1}{\gamma_1(1-\theta^0) + \gamma_2 \theta^0} \theta^0 (1-\theta^0). \tag{5}$$

Because our model is static and we have $\hat{x} = 0$, and because the size of uncertainty, ε, does not in any other way affect the portfolio solution under the DS method, there is a strong implication: it turns out that in our two period model also higher-order approximations, up to any order, are identical to the constant zero-order part of the solution, $\bar{\alpha}_e$. The DS portfolio solution for θ is then obtained as:

$$\theta = \theta^0 + \frac{\alpha_e}{p}, \text{ where } \alpha_e = \overline{\alpha}_e. \tag{6}$$

The property of α_e which is key here, is that it is invariant to the size, or any other statistical properties (i.e. skewness, kurtosis etc.), of the shock z in the model. It should be clear that this is true in our model from inspecting equation (5) – α_e only depends on the difference between the two investors' risk aversion parameters and the initial equity endowments.[6].

2.3 Judd-Guu portfolio solution

To obtain the portfolio solution using bifurcation methods we closely follow the steps outlined in Judd and Guu (2001). To save space, we do not repeat them here and refer to appendix C or the original paper for further documentation. We only point out that, unlike the DS method, Judd and Guu's method produces a solution which depends on the size of uncertainty in the model, and the relationship between the size of uncertainty and portfolio solution depends on higher-order moments of assets' returns. Namely, the first-order terms of JG's approximate solution depend on the returns' skewness, while the second-order terms depend on their kurtosis.

[5]That is, both countries' Euler equations with respect to the risky and with respect to the safe asset.

[6]Strictly speaking, the finding that α_e is invariant to changes in the size of uncertainty does not imply that the same is true for θ, as the equity price, p, generally does depend on ε. In appendix B, we show that taking into the account the effect of the size of shocks on p would, in fact, worsen the performance of the DS solution for θ.

3 Results

In this section we document our results for some quantitative examples of the model economy. We consider a setup in which both countries have identical initial endowments, such that $b_i^0 = 0$ and $\theta_i^0 = 0.5$ for country $i = 1, 2$, but in which country 2 is more risk averse than country 1, reflected by coefficients of risk aversions $\gamma_1 = \gamma_2/2$. We also need to choose the distributional assumptions of the world output endowment, $Y = 1 + \varepsilon z$, which determine the moments of equity returns in our model economy – in excess over the returns on the safe asset. It is reasonable to expect that local portfolio solutions suffer in accuracy as the size of uncertainty, ε, gets bigger. To judge whether this happens at economically relevant parameterizations, we should, in our numerical examples, take empirically observed distributions on (excess) equity returns, and the robust empirical stylized fact of positive equity premia, seriously. It is well known that (excess) equity returns are not normally distributed. Guidolin and Timmermann (2008) provide detailed stylized facts on the first four moments of excess returns of several aggregate stock market indices, based on monthly MSCI indices for the Pacific ex-Japan, United Kingdom, United States, Japan, Europe ex-UK, and World – we repeat them in the first four columns of Table 1. We use Guidolin and Timmermann's reported empirical moments to calibrate our endowment process. We do so by using a Normal-inverse Gaussian (N.I.G.) distribution, which gives enough flexibility to target mean, standard deviation, skewness and kurtosis of equity returns in our model.[7] In particular, for each MSCI index we consider, we choose 4 parameters of the N.I.G. distribution to make sure that $E\left\{z^3\right\}$ and $E\left\{z^4\right\}$ match the observed skewness and kurtosis of that MSCI index' returns from the data, and that $E\{z\} = 0$ and $E\left\{z^2\right\} = 1$ (which is the normalization assumed by Judd and Guu (2001), which we follow here). Since we set $E\left\{z^2\right\} = 1$, we control the volatility of the return process through the choice of ε.[8] The resulting parameters that characterize the distribution of z are reported in table 2 in appendix A. Finally, we pick our final free parameter, γ_2, to match the observed mean excess equity return.

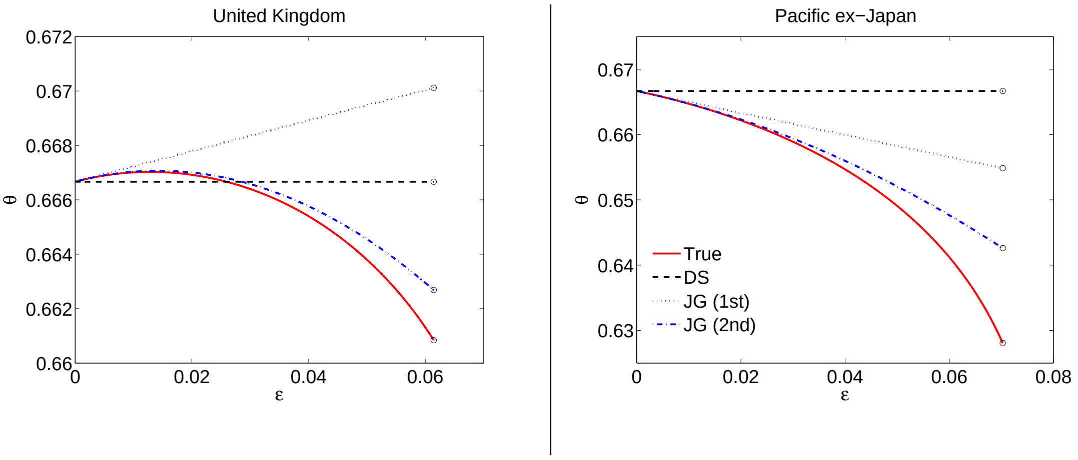
Figure 1 plots the portfolio solution for country 1's equity share, θ, for the above parameterization, when the model's returns have been calibrated to the particular cases of MSCI United Kingdom, and MSCI Pacific ex-Japan; we choose these two indices because we think those are well suited to illustrate some of the properties of the portfolio solution methods (namely the impact of positive, in case of UK, and negative, in case of Pacific ex-Japan, skewness, and substantial kurtosis on the equilibrium portfolios). In each of the panels, we plot the portfolio solutions as a function of ε, to illustrate how the solutions depend on the assumed size of uncertainty. The endpoints of the lines, marked by 'circle'-signs, correspond to the actual calibrated values of ε, such that the model's volatility of excess equity return equals its respective index's standard deviation.

Let us focus on the results for the 'United Kingdom', in the first panel of Figure 1. This particular region's MSCI return index displays positive skewness (0.75) and substantial kurtosis (10.3). The solid red line displays the true portfolio solution. As country 1 is less

[7]The N.I.G. distribution, together with the skew normal distribution or the skew-student-t distribution, belongs to the class of distributions that have experienced recent interest in the finance literature because of its flexibility in capturing non-normal properties of asset pricing data (see e.g. Colacito et al. (2012)).

[8]In our model the variance of gross equity return, R_e, is given by $var(R_e) = var\left(\frac{1+\varepsilon z}{p}\right) = \frac{\varepsilon^2}{p^2} = \varepsilon^2\left[E\left(R_e\right)\right]^2$, because $E[z^2] = 1$ and $[E\left(R_e\right)] = 1/p$. Using this result, we set $\varepsilon = \frac{std(r_e^{data})}{\left[E\left(r_e^{data}\right)+1\right]}$, where r_e^{data} is the net return in the data.

Figure 1: Equity shares held by country 1 investor. Panel A and B refer to the parameterizations for the UK and Pacific stock market facts respectively. Circles correspond to the value of ε used in the calibration.



risk averse, it chooses to hold a higher share of equity than it is initially endowed with ($\theta > \theta^e = 0.5$), which it finances by going short in debt. As ε increases, we observe that country 1's optimal share in equity initially increases and then, at a certain size of uncertainty, starts to decrease. The portfolio solution obtained by the Judd-Guu approach can help us understand the mechanisms that drive these results in more detail.

As expected, the JG solution converges to the true solution as ε goes to 0. The positive skewness of MSCI UK returns leads to a positive slope of the first-order (linear) Judd-Guu solution; this suggests that, up to first order, positive skewness tends to increase country 1's optimal equity holdings, θ, as the size of uncertainty increases. To understand this finding, notice that positive skewness means shifting more weight to 'good' outcomes, such that an investor would demand more of the risky asset. This logic, however, applies to both investors: investor of country 1, but also to the investor of country 2 demand more of the risky asset as a result of positively skewed returns. JG show that the strength with which equity demand increases in such case depends on investors' relative 'skew tolerance'. For the CRRA preference specification we use, skew tolerance is always larger for the less risk-averse country, implying that country 1's appetite for taking risk increases more strongly and its chosen equity position goes up under positive skewness as ε increases.[9] To understand the effects of (excess) kurtosis on the optimal portfolio holdings it is instructive to look at the second-order JG solution, and to understand the second-order component of JG's solution. Kurtosis means putting more weight to tail events, so as the size of uncertainty increases, this leads an investor to reduce demand for the risky asset. Again, this logic applies to both investors. The strength by which investors want to reduce their holdings of the risky asset depends on investors' relative 'kurtosis tolerance'. For CRRA preferences kurtosis tolerance is lower for the less risk-averse

[9] Judd and Guu (2001) define 'skew tolerance' as $\rho(C_i) = \frac{1}{2}\frac{u'(C_i)}{u''(C_i)}\frac{u'''(C_i)}{u''(C_i)}$, for country $i = 1, 2$. For CRRA preferences this is given by $\rho(C_i) = \frac{1}{2}\frac{\gamma_i + 1}{\gamma_i}$. Note that in this case $\frac{\partial \rho}{\partial \gamma_i} = -\frac{1}{2\gamma_i^2} < 0$. Therefore, with $\gamma_1 < \gamma_2$ we have that $\rho(C_1) > \rho(C_2)$.

country[10], so that the reduction in the demand for risky assets due to (excess) kurtosis is more pronounced for the less risk-averse country. The second-order component of the JG solution, incorporating the effects from kurtosis, therefore decreases the optimal share of equity as ε increases, moving it closer to the true solution. Further improvements in the JG solution could be expected, when higher order components were added.

Panel 2 of Figure 1, 'Pacific ex-Japan', provides a different example. The MSCI Pacific ex-Japan return index is again characterized by substantial, even higher kurtosis (22.3), but in contrast to the previous example displays negative skewness (-2.3). The negative skewness implies that the return distribution is more heavily shifted towards 'bad' outcomes, as a result of which investors would demand less of a risky asset. Since the skew-tolerance coefficient continues to be higher for country 1, but now, because of negative skewness, multiplies a negative number $E\left(z^3\right)$, the slope from the first-order part of the JG solution is negative: the less risk averse country 1 decreases its holdings of the risky asset as the size of uncertainty becomes larger. The effect of (excess) kurtosis work as in the above example of the UK, but are quantitatively more pronounced in the case of the even higher kurtosis of the MSCI Pacific. The second-order JG solution again captures this effect pretty well.

Finally, the black dashed line in Figure 1 shows the results from applying the DS solution method. The DS solution coincides with the constant (zero-order) component of the Judd-Guu solution. As explained in section 2, by construction of the DS method, the portfolio solution under DS is a function of state variables only, and not a direct function of the size of risk, ε. Since, in this simple static model there is no variation in states, the obtained constant solution is not only the zero-order solution, but actually corresponds to the DS solution up to any order.

Table 1 reports the optimal portfolio solutions for all other regions, calibrated to the respective MSCI return indices. Columns 5-8 show the solutions for the scenario in which country 2 is twice as risk averse as country 1. They report – for various calibrations to the moments of the respective MSCI indices – the true portfolio solution, the (second-order) JG solution, and the DS solution. The largest discrepancies between solution methods emerge when the model is calibrated to the moments of MSCI Pacific ex-Japan: the difference to the true solution of the equity share obtained by the (second-order) JG solution is -2.31%, the difference of the DS solution -6.13%. The results from the θ^{DS}-column again illustrate that the DS solution is not affected by the size of uncertainty or any other distributional features of the disturbances.

We also perform some sensitivity analysis and report the portfolio solutions for a scenario in which country 2 is three times as risk averse as country 1. In this case, for the MSCI Pacific ex-Japan calibration, the differences from the true optimal equity holdings rise to -3.14% for the (second-order) JG solution, and to -7.56% for the DS solution.

[10] JG's definition of 'kurtosis tolerance' is gives by $\kappa\left(C_i\right) = -\frac{1}{3}\frac{u''''\left(C_i\right)}{u''\left(C_i\right)}\frac{u'\left(C_i\right)}{u''\left(C_i\right)}\frac{u'\left(C_i\right)}{u''\left(C_i\right)}$. For CRRA preferences, $\kappa\left(C_i\right) = -\frac{1}{3}\frac{(\gamma_i+1)(\gamma_i+2)}{\gamma_i^2}$. Note that in this case $\frac{\partial\kappa}{\partial\gamma_i} = \frac{\gamma_i+2}{\gamma_i^3} > 0$. Therefore, with $\gamma_1 < \gamma_2$ we have $\kappa\left(C_1\right) < \kappa\left(C_2\right)$.

Asset	Data				$\gamma_2/\gamma_1 = 2$			$\gamma_2/\gamma_1 = 3$		
	Mean,%	SD,%	Skew	Kurt	θ^{DS}	θ^{JG}	θ^{true}	θ^{DS}	θ^{JD}	θ^{true}
UK	0.7503	6.1898	0.7587	10.316	0.6667	0.6626	0.6608	0.7500	0.7449	0.7417
Pacific Ex-Japan	0.3892	7.0538	-2.2723	22.297	0.6667	0.6427	0.6282	0.7500	0.7195	0.6973
World	0.4560	5.174	-0.8711	6.9133	0.6667	0.6607	0.6588	0.7500	0.7424	0.7396
US	0.5415	4.4825	-0.7084	5.9138	0.6667	0.6623	0.6607	0.7500	0.7445	0.7421
Japan	0.3733	6.4830	0.0700	3.5044	0.6667	0.6653	0.6642	0.7500	0.7483	0.7466
Europe ex-UK	0.4158	5.0578	-0.5672	4.6124	0.6667	0.6631	0.6620	0.7500	0.7454	0.7439

Table 1: Optimal equity holdings obtained by different portfolio solution methods; model calibrated to (various regions') return data on MSCI aggregate stock market indices by Guidolin and Timmerman (2008).

4 Conclusions and Future Research

Using a stylized two period model we obtain portfolio solutions from two solution approaches that belong to the class of local approximation methods – the approach of Judd and Guu (2001) and the approach of Devereux and Sutherland (2011, 2010) – and compare them with the true portfolio solution. We show that in the true solution the size of uncertainty affects the solution, and the precise way in which it does so depends on the distributional properties of equity returns (skewness, kurtosis, as well as higher moments). The DS method, while providing an accurate solution in the limit as uncertainty goes to zero, fails to capture these determinants of the portfolio solution, and stays unaffected by variations in the size of uncertainty or the shape of the distribution. The JG solution goes a step further in approximating the true solution – the JG solution generally varies as the size of uncertainty changes, and already a second-order JG solution can account for effects of skewness and kurtosis of equity returns.

The two period model of the present paper has allowed us to shed light on some of the properties of the local approximation methods of DS and JG to solving portfolio problems. We should note, however, that the advantages of the DS solution approach are downplayed in such a two period setting. They lie in obtaining portfolio solutions in a *dynamic* setting, possibly in environments in which there are many states variables and in which global approximation methods for finding portfolio solutions are computationally expensive or infeasible. We explore the performance of the DS portfolio solution method in more general, dynamic settings in a companion paper, Rabitsch et al. (2013), and contrast it to globally approximated portfolio solutions. The JG bifurcation approach, while, compared to DS, delivering much more accurate solutions in our two period model economy, has yet to be extended in order to be applied to finding portfolio solutions in dynamic models.

A Details of the Nonlinear (Quadrature) Solution

The key step in obtaining the quadrature solution is to replace the integrals in (1) with finite sums. We do so by using the Gauss-Chebyshev quadrature. We assume that z follows a truncated normal inverse Gaussian distribution (NIG). The NIG distribution is completely characterized by 4 parameters (μ_{nig}, α_{nig}, β_{nig} and δ_{nig}). This allows us to match the first 4 moments of the returns from the data. In addition, we assume that the support of z is bounded from below, $z >= \underline{Z}$, so that $Y > 0$ for all values of ε that we consider. In practice,

Asset	NIG parameters						
	μ	α	β	δ	ε	γ_2^*	γ_2^{**}
Pacific Ex-Japan	0.1439	0.4163	−0.1745	0.3114	0.0703	0.886	1.120
UK	−0.1138	0.6932	0.1171	0.6638	0.0614	2.969	3.920
World	0.2903	1.0839	−0.3176	0.9473	0.0515	2.344	3.096
US	0.3104	1.2331	−0.3352	1.0990	0.0446	3.750	4.950
Japan	−0.1406	2.4628	0.1411	2.4507	0.0646	2.294	3.051
Europe ex-UK	0.4776	1.7463	−0.5250	1.5150	0.0504	2.320	3.079

*– for the case $\frac{\gamma_2}{\gamma_1} = 2$, **– for the case $\frac{\gamma_2}{\gamma_1} = 3$

Table 2: Calibrated parameter values

we assume that $\underline{Z} = -10$ in all cases, except for when we consider MSCI Pacific ex-Japan with $\gamma_2 = 3\gamma_1$, where we assume that $\underline{Z} = -9$. This ensures that $C_1 > 0$ and $C_2 > 0$ for all values of ε that we consider. After fixing $\underline{Z}$ and some large upper bound $\bar{Z}$[11], we set the values for μ_{nig}, α_{nig}, β_{nig} and δ_{nig}, apply the Gauss-Chebyshev quadrature with 1000 nodes[12] to compute the resulting first 4 moments, and change values of μ_{nig}, α_{nig}, β_{nig} and δ_{nig} until we obtain $E[z] = 0$, $E[z^2] = 1$, and $E[z^3]$ and $E[z^4]$ that match the skewness and kurtosis of assets' returns in the data.

After this, we solve the system in (1) with a non-linear solver on a fine grid over $[\underline{\varepsilon}, \bar{\varepsilon}_i]$, where $\bar{\varepsilon}_i$ corresponds to the standard deviation of the asset i's returns in the data.

B Details of the Devereux-Sutherland Solution

We can re-write home investor's budget constraints as:

$$0 = \underbrace{(\theta - \theta^0)p}_{\alpha_e} + \underbrace{\left(b - b^0\right)}_{\alpha_b} = \alpha_e + \alpha_b = W$$

$$C_1 = (\theta - \theta^0)p \underbrace{\frac{Y}{p}}_{=R_e} + \left(b - b^0\right) \underbrace{1}_{=R_b} + b^0 + \theta^0 Y = \alpha_e R_e + \alpha_b R_b + b^0 + \theta^0 Y$$

$$= W R_b + \alpha_e \underbrace{(R_e - R_b)}_{=R_x} + b^0 + \theta^0 Y$$

Since the first equation implies that $W = 0$, the equilibrium system can be written as:

$$\begin{array}{ll}
\text{(E1'): } \lambda_1 = E\left[u_1'(C_1) R_b\right], & \text{(E2'): } \lambda_2 = E\left[u_2'(C_2 R_b)\right], \\
\text{(E3'): } \lambda_1 = E\left[u_1'(C_1) R_e\right], & \text{(E4'): } \lambda_2 = E\left[u_2'(C_2) R_e\right], \\
\text{(E5'): } C_1 = \alpha_e R_x + b^0 + \theta^0 Y, \forall Y, & \text{(E6'): } C_1 + C_2 = Y, \forall Y, \\
\text{(E7'): } \theta^0 p + b^0 = \theta p + b, &
\end{array}$$

Following Devereux and Sutherland we take a second order approximation to the Euler equations w.r.t. to equity and w.r.t. to the bond, and combine the resulting expressions for each country. This gives:

[11] In practice, we set $\bar{Z} = 30$, and check that the results are not sensitive to changing this value.

[12] We check that the results are not sensitive the the number of quadrature nodes selected as well.

$$
\begin{aligned}
\bar{C}_1^{-\gamma_1}\bar{R}_b \cdot E\left[\hat{r}_x - \gamma_1\hat{c}_1\hat{r}_x + \frac{1}{2}(\hat{r}_e^2 - \hat{r}_b^2)\right] &= 0 \\
\bar{C}_2^{-\gamma_2}\bar{R}_b \cdot E\left[\hat{r}_x - \gamma_2\hat{c}_2\hat{r}_x + \frac{1}{2}(\hat{r}_e^2 - \hat{r}_b^2)\right] &= 0
\end{aligned}
$$

Combining, we get:

$$
E\left[\left(\gamma_1\hat{c}_1 - \gamma_2\hat{c}_2\right)\hat{r}_x\right] = 0 \tag{7}
$$

That is, we need first order expressions for consumptions of country 1 and 2, and of excess returns. Those are found by log-linearizing (E5'), (E6') and the definition of excess returns, $R_e = Y/p$, and substituting the $\bar{\alpha}\hat{r}_x$ term with a mean-zero shock ξ in (E5'):

$$
\begin{aligned}
\bar{C}_1\hat{c}_1 &= \xi_1 + \theta_0\bar{Y}\hat{y}_1, \\
\bar{C}_2\hat{c}_2 &= \bar{Y}\hat{y} - \bar{C}_1\hat{c}_1, \\
\hat{r}_x &= \bar{Y}\hat{y}
\end{aligned}
$$

Plugging the above expressions for $\hat{c}_1$, $\hat{c}_1$ and $\hat{r}_x$ into equation (7), using the fact that $\bar{C}_1 = \theta^0\overline{Y}$ and $\bar{C}_2 = \left(1-\theta^0\right)\overline{Y}$ and plugging back $\alpha_e\hat{r}_x$ for ξ, we get:

$$
\left(\frac{\gamma_1(\alpha_e+\theta^0)}{\theta^0} - \frac{\gamma_2(1-\theta^0-\alpha_e)}{1-\theta^0}\right)E\hat{y}_1^2 = \left(\frac{\gamma_1(\alpha_e+\theta^0)}{\theta^0} - \frac{\gamma_2(1-\theta^0-\alpha_e)}{1-\theta^0}\right)\varepsilon^2 = 0
$$

where we used $\hat{y} = \varepsilon z$ and $Ez^2 = 1$.

Solving the last equation for α_e, we get:

$$
\alpha_e = \frac{\gamma_2 - \gamma_1}{\gamma_1(1-\theta^0) + \gamma_2\theta^0}\theta^0(1-\theta^0) \tag{8}
$$

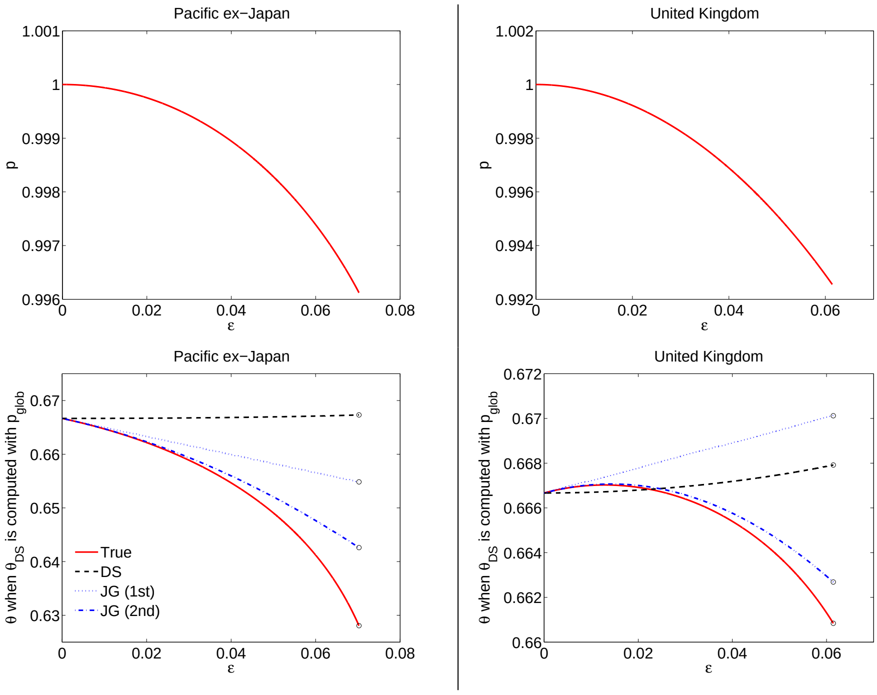
Once the optimal α_e is found, the solution to θ can be found from $\theta = \theta^0 + \frac{\alpha_e}{p}$.

While from equation (8) it is clear that α_e does not depend on the size of shocks, ε, this is not generally true for θ, as p in general will depend on ε in higher-order approximations. To see, how the portfolio solutions from the DS method would perform if one accounted for this, we use the solution for p from the true portfolio solution method. The idea is, that at best, an infinite-order Taylor approximation would converge to the true function $p(\varepsilon)$. As the first row of Figure 2 shows, $p(\varepsilon)$ is, however, a decreasing function of ε (the return on the risky asset increases as the size of shocks increases, so its price falls). This implies, that allowing p to vary with ε would actually worsen the portfolio solution results from the DS method, which is confirmed in the second row of Figure 2; θ^{DS} increases as ε increases.

C Details of the Judd-Guu Solution

The system in (1) implicitly defines $\theta(\varepsilon)$ and $p(\varepsilon)$. Denote this system $H(\theta(\varepsilon), p(\varepsilon), \varepsilon) = 0$. However, the implicit function theorem cannot be applied to analyze (1) around $\varepsilon = 0$, since assets are perfect substitutes in such case and must trade at the same price; that is, we must have $p(0) = 1$. However, $\theta(0)$ is indeterminate because $H(\theta, p, 0) = 0$ for all θ. The

Figure 2: Equity price from true solution, p^{true}, and solutions for equity shares held by country 1 investor.



indeterminacy of θ implies that $H_\theta(\theta, 1, 0) = 0$, ruling out application of the implicit function theorem.

Judd and Guu (2001) show how one can use the bifurcation theorem to solve the above problem. The bifurcation approach requires that the Jabobian matrix $H_{(\theta,p)}$ is a zero matrix. While at $\varepsilon = 0$, $\theta(0)$ is indeterminate, there is only a single possible value for $p(0)$ and $p'(0)$; so the Jacobian $H_{(\theta,p)}$ would in fact not be a zero matrix. We follow Judd and Guu (2001) in solving this problem by reformulating the problem in terms of the price of risk, π, instead of the price of equity, p. That is, we parameterize the equity price as $p\varepsilon = 1 - \varepsilon^2\pi(\varepsilon)$, where $\pi(\varepsilon)$ is the risk premium in the ε-economy. Since $\sigma_z^2 = 1$, ε^2 is the variance of risk and $\pi(\varepsilon)$ is the risk premium per unit variance. This way, the system in (1) can be rewritten as

$$\mathbf{H}(\theta(\varepsilon), \pi(\varepsilon), \varepsilon) = \tag{9}$$

$$\begin{bmatrix} E\left[u_1'(\theta(1+\varepsilon z) + b^e + (\theta_0 - \theta)\left[1-\varepsilon^2\pi\right])(z+\varepsilon\pi)\right] \\ E\left[u_2'((1-\theta)(1+\varepsilon z) - b^e - (\theta_0 - \theta)\left[1-\varepsilon^2\pi\right])(z+\varepsilon\pi)\right] \end{bmatrix} = \mathbf{0}.$$

Obtaining the coefficients of the Taylor series expansion of $\theta(\varepsilon)$, given by

$$\theta(\varepsilon) = \theta_0 + \theta'(0)\varepsilon + \theta''(0)\frac{\varepsilon^2}{2} + \theta'''(0)\frac{\varepsilon^3}{6} + ..., \tag{10}$$

is then conceptually straightforward. To find θ_0, one needs to differentiate function $\mathbf{H}$ with respect to ε, to find $\theta'(0)$ one needs to differentiate function $\mathbf{H}$ w.r.t. ε the second time, to find $\theta''(0)$ the third time, etc., and needs to evaluate those derivatives at $\varepsilon = 0$.

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