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Border Collision Bifurcations in Boom and Bust Cycles

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Abstract.

Boom and bust cycles are widely documented in the literature on industry dynamics. Rigidities and delays in capacity adjustment in combination with bounded rational behavior have been identified as central driving forces. We construct a model that features only these two elements and we show that this is indeed sufficient to reproduce some stylized facts of a boom and bust cycle. The bifurcation diagrams summarizing the dynamic behavior reveal complex cycles and in particular also abrupt changes in the nature of these cycles. We apply new insights from the mathematical theory of piecewise smooth dynamic systems - in particular, results from the theory of border collision bifurcations - and show that the very existence of borders such as capacity constraints or nonnegativity constraints may lie behind abrupt changes in the dynamic behavior of economic variables.

Keywords: Boom and bust cycle, cobweb dynamics, piecewise smooth system, border collision bifurcation.
JEL classification: B52, C61, C62, C63, D41

1 Introduction

Boom and bust cycles - during which an increase in demand is accompanied by an even stronger increase in capacity, leading to overcapacity and to a subsequent decline of the industry - are widely documented in business literature. In their overview Dosi et al. (2008, with reference to Paich and Sterman 1993; Sterman 2000; Sterman et al. 2007) mention examples in durable consumer electronics (e.g. televisions, VCR's, calculators, etc.), telecommunications, medical equipment, chemicals, real estate, pulp and paper, agricultural commodities, natural resources, toys and games, tennis equipment, bicycles, semiconductors and running shoes.

In a seminal paper studying the implications of different firm strategies for the industry dynamics, Sterman et al. (2007) start from a rich set of behavioral assumptions and construct an analytically formulated, highly complex nonlinear model; too complex in fact for obtaining analytic solutions. Using computer simulations they identify the combination of boundedly rational managerial behavior with rigidities and delays in capacity adjustments as crucial for the occurrence and the nature of boom and bust cycles. Their modeling of managerial behavior involves the use of anchoring and adjustment heuristics, the use of simple forecasting rules on the basis of past observations, and a differentiation between an "aggressive" and a "cautious" type of behavior. Sterman et al. (2007) and in particular also Dosi et al. (2008) provide ample evidence from experiments and case studies that boundedly rational behavior of this type is quite persistent – and the scope for learning is very limited – in complex, nonlinear dynamic situations such as boom and bust cycles.

In the present paper, we consider a model as simple as possible that retains the two central aspects identified by Sterman et al. (2007), namely boundedly rational behavior on the one hand and, in particular, delays and rigidities in capacity adjustments on the other hand. It is on the role of the latter that the focus of our paper rests; and this has three implications for the model: First, we explicitly consider a gestation lag for increasing capacity; second, we explicitly take into account that downward adjustment of capacity is constrained by the depreciation rate (reflecting the assumption that machinery once installed is specific to the particular industry); and third, we explicitly take into account that output expansion is constraint by existing productive capacity. We combine these aspects of non-instantaneous capacity adjustment with a boundedly rational behavior similar to the type used in Sterman et al. (2007) and corroborated by Dosi et al. (2008); in particular, we use anchor and adjustment heuristics and simple expectation formation hypotheses. Finally, to keep the model as simple as possible, we abstract from strategic interactions and assume a fully competitive market. From an analytic point of view, it should be noted that the model will be specified in discrete time: This allows a simple representation delays; more importantly and already noted by Saari (1985) a continuous time formulation reduces by assumption the possibility of "overreaction" and instability.

The role of capacity constraints, on which the focus of our paper rests, is also discussed in the literature on Hicksian growth and business cycle models (for a recent contribution see: Shusko et al., 2010) and in oligopoly literature (for recent contributions see e.g. Besanko and Doralszelki, 2004; Lu and Poddar, 2005; Tramontana et al., 2009). However, assuming a perfectly competitive market structure, our model is in the tradition of Cobweb models. In this field, recent contributions analyze primarily the role of learning and of heterogeneous agents (for an overview see: Hommes 2008; and for a recent example: Caulkins and Baker 2010); some papers studied the role of buffer stocks (see e.g.: Athanasioua et al., 2008); and the role of demand or supply linkages (see Currie and Kubin, 1995; Dieci and Westerhof, 2010). The role of capacity constraints and capacity adjustments got little attention although its importance is explicitly stressed e.g. for agricultural markets (see Gouel 2011; Declerck and Cloutier 2010). The only exception to our knowledge is Currie and Kubin (1997), on which we base our subsequent analysis.

The simplicity of the model – while admittedly losing many an empirical relevant aspect – allows studying the interplay of the two driving forces quasi in insulation and it allows applying recent analytic results from the theory of nonlinear dynamic systems. We show that the model can reproduce some stylized facts of boom and bust cycles and, in particular, that the dynamic behavior and the nature of theses cycles may change abruptly. Applying the new analytic insights provides an economic interpretation of these abrupt changes. The model is ultimately described by a continuous two dimensional map. Taking delays and rigidities in capacity adjustment explicitly into account introduces borders – that have an economics' motivation – into the phase space at which the definition of the dynamic system changes. This type of systems, called piecewise smooth, exhibits properties which are similar to those of smooth systems; but, in addition, new phenomena may occur: The possibility that a border is crossed at which the definition of the map changes leads to a new type of bifurcations, called border collision bifurcations (following Nusse and Yorke 1992, 1995). To be more precise: This type of bifurcation occurs if some attracting set collides with a border. Due to the fact that applied models often are only piecewise smooth, the study of these particular bifurcations has been considerably improved in recent times, and since a few years, border collision bifurcations are noted also in economics' models (see Agliari et al., 2011; Gardini et al. 2008, 2011; Tramontana et al. 2009, 2010, 2011 to cite a few). However, this is still a novel argument, and many studies of piecewise smooth systems actually only perform a local stability analysis of the fixed point(s) and illustrate the global dynamics by use of simulations, via bifurcation diagrams, in order to evidence abrupt changes in the shape or cyclicity of an attractor or an abrupt change from a cyclical to a chaotic attractor. Typically – and also in Currie and Kubin (1997) – no economic explanation is offered for these abrupt transitions. In the following we show that some of them involve border collision bifurcations. Therefore, those abrupt transitions are not only a mathematical curiosity connected to the piecewise smooth specification of the system, but are intimately related to borders the existence of which have a genuine economics' rational. We consider this type of bifurcation of particular importance for economics models because there exist many borders with an explicit and genuine economics motivation, such as e.g. capacity constraints and nonnegativity constraints. The importance of such constraints for economic models is underlined by the possibility of border collision bifurcations that engender an abrupt change in the dynamic pattern.

After this introduction, the rest of the paper is as follows. In Sec. 2 we recall the model, which was originally proposed in Currie and Kubin (1997), illustrating how the constraints follow from natural economic assumptions, and how this complicates the response of the economic model, depending on the different values that the state vector can assume. In the same section the proof of the uniqueness of the equilibrium is given, and its local stability analysis is recalled. In Sec. 3 we analyze the global dynamics. We first illustrate that the model can reproduce stylized facts of boom and bust cycles. We then describe the cyclical pattern of the global dynamics occurring in the region in which the equilibrium is unstable. Closed attracting curves with cyclical behavior change abruptly with stable cycles of different periods and chaotic attractors. While in smooth systems the changes in the attracting cycles can be often predicted (they occur if an eigenvalue is equal to 1 in absolute value), in the case of piecewise smooth systems this becomes more difficult. For example, the eigenvalues of the cycles play no role in the border collision bifurcations. Instead, we may expect a border collision bifurcation when a periodic point of a cycle is close to a border defining a region of the map, when it "collides" with the border. After a border collision bifrucation it is difficult to predict what will be the attracting set (for example it may a different cycle, cyclical chaotic pieces, or a chaotic attractor in one piece). Moreover, the occurrence of border collision bifurcations often leads to multistability, between cycles and chaotic attractors or between two different cycles. The coexistence also reveals another kind of unpredictability due to the fractal structure of the related basins of attraction. An uncertainty in the position of the phase space, or a shock, may cause the transition from convergence to one attractor to convergence to a different one.

In Section 3 we thus show that the very existence of economically motivated borders leads to the possibility of abrupt and drastic changes in the cyclical behavior of economic variables. Section 4 concludes.

2 The basic framework

As in Currie and Kubin (1997), we study the dynamics of a competitive market with a homogenous commodity. Technology is assumed to be as simple as possible: Production takes time (one period); and requires as input one unit of fixed capital (one machine) and a constant quantity of labor and raw material per unit of the final output. While labor and raw material can be acquired instantaneously at a fixed price, denoted by ν, the stock of machines can only be increased with a gestation lag of one period. It depreciates irrespective of use at a constant rate δ per period. The price of one new machine, denoted by m, is constant. Once installed, machines are specific to the particular industry and thus their use involves no opportunity costs.

We do not allow for any (buffer) stocks of the final commodity and the entire final output, denoted by q_t, is sold at a market clearing price. The market demand function is assumed to be linear and subject to the following constraints:

$$p_t = \begin{cases} 0 & \text{for } q_t = 0 \\ a - q_t & \text{for } 0 < q_t \leq a \\ 0 & \text{for } q_t > a \end{cases} \tag{1}$$

where a denotes the maximum quantity that can be sold at a positive price and where we assume – for completeness - that with a zero quantity the price is also zero (since no price can be observed).

For modeling entrepreneurial behavior, we start from the following observation: In a competitive market with many (homogeneous) competitors a single entrepreneur would have difficulties to observe total market capacity, total market supply and total market demand, but she can easily observe her own capacity, her own quantity produced and the market price for the output. This leads into a Marshallian tradition that we combine with managerial decision rules involving anchoring and adjustment heuristics. The individual quantities are used as anchor and they are adjusted – following a Marshallian tradition — whenever the current market price is not equal to the relevant Marshallian supply price.

Given that the costs of the inherited machines are sunk, a Marshallian short-period supply price is equal to the effective variable cost $\nu(1+i)$ where i denotes the (fixed) interest rate for financing the variable inputs over the production period. We assume that producers follow an adjustment rule whereby, subject to the capacity constraint, they increase their anchor variable output if the current market price is less than this supply price. Specifically

$$\frac{q_{t+1}-q_t}{q_t} = \mu(p_t - \nu(1+i)) \text{ subject to } 0 \leq q_{t+1} \leq k_t \tag{2}$$

otherwise – and this is the first important constraint - it is assumed $q_{t+1} = k_t$. Here p_t is given in (1), $\mu > 0$ denotes the speed for output adjustment, and k_t indicates the current stock of machines, i.e. the current capacity. The equation in (2) can be rewritten as follows: since $q_{t+1} = (1 - \mu\nu(1+i))q_t + \mu p_t q_t = (1 + \mu(p_t - \nu(1+i)))q_t$, we have

$$q_{t+1} = \begin{cases} (1+\mu(p_t - \nu(1+i)))q_t & \text{if } \ 0 \leq (1+\mu(p_t - \nu(1+i)))q_t \leq k_t \\ k_t & \text{if } \ (1+\mu(p_t - \nu(1+i)))q_t > k_t \end{cases} \tag{3}$$

For the capacity adjustment decision, the long-period supply price, denoted by c, is given by

$$c = \nu(1+i) + m(\delta + i) \tag{4}$$

where $m(\delta + i)$ represents the ex ante per period costs of buying and installing a machine instead of investing in the financial market.

Entrepreneurs increase capacity whenever the current (or going) price is higher than the long-period supply price according to the following rule:

$$\frac{k_{t+1}-k_t}{k_t} = \sigma(p_t - c) \text{ subject to } \frac{k_{t+1}-k_t}{k_t} \geq -\delta \tag{5}$$

where p_t is given in (1), $\sigma > 0$ denotes the speed for capacity adjustment, and where the lower bound, equivalent to $k_{t+1} \geq (1-\delta)k_t$, reflects the inability of producers to sell second-hand machines. If the constraint is not satisfied then it is assumed $\frac{k_{t+1}-k_t}{k_t} = -\delta$, that is $k_{t+1} = (1-\delta)k_t$. This is the second important constraint. So the capacity is defined via

$$k_{t+1} = \begin{cases} (1+\sigma(p_t - c))k_t & \text{if } \sigma(p_t - c) \geq -\delta \\ (1-\delta)k_t & \text{if } \sigma(p_t - c) < -\delta \end{cases} \tag{6}$$

The economic model is represented by the equations in (3) and (6), where p_t is given in (1). Thus it is ultimately described by a two-dimensional continuous map in the plane, say $(q_{t+1}, k_{t+1}) = T(q_t, k_t)$, where T is not smooth, as its definition changes (although continuously) crossing some borders (in the phase plane (q, k)) due to the constraints.

By inserting (1) into (3) we have the following output dynamics:

$$q_{t+1} = \begin{cases} f_1(q_t) = (1 + \mu(a - \nu(1+i)))q_t - \mu q_t^2 & \text{if } \; 0 < q_t \le a \; \text{ and } \; k_t \ge f_1(q_t) \\ f_2(k_t) = k_t & \text{if } \; 0 < q_t \le a \; \text{ and } \; k_t < f_1(q_t) \\ f_3(q_t) = (1 - \mu\nu(1+i))q_t & \text{if } \; q_t > a \; \text{ and } \; k_t \ge f_3(q_t) \\ f_2(k_t) = k_t & \text{if } \; q_t > a \; \text{ and } \; k_t < f_3(q_t) \end{cases} \tag{7}$$

Inserting (1) into (6) gives the following capacity dynamics:

$$k_{t+1} = \begin{cases} g_1(q_t, k_t) = (1 + \sigma(a - c - q_t))k_t & \text{if } \; q_t \le \min(a^*, a) \\ g_2(k_t) = (1 - \xi)k_t & \text{if } \; q_t > \min(a^*, a) \end{cases} \tag{8}$$

where

$$a^* = a - c + \frac{\delta}{\sigma} \tag{9}$$

and the contraction factor $(1 - \xi)$ is computed via

$$\begin{aligned} \xi &= \sigma c \;\; \text{if } \; a \le a^* \text{ (i.e. if } \sigma c \le \delta) \\ \xi &= \delta \;\;\; \text{if } \; a > a^* \text{ (i.e. if } \sigma c > \delta) \end{aligned} \tag{10}$$

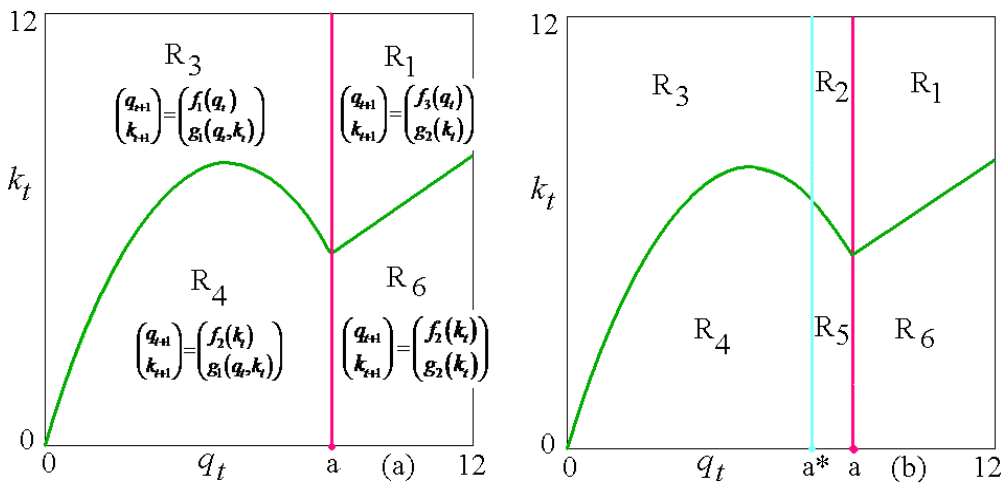

Fig.1 Regions in the phase space. In (a) the case with $a^* < a$ (i.e. when $\sigma c > \delta$). In (b) the case $a \le a^*$ (i.e. when $\sigma c \le \delta$).

Summarizing, the phase space can have several regions, where the map T takes different definitions. For parameters values such that $\sigma c \le \delta$ the phase space consists in four regions (see Fig.1a), separated the vertical straight line of equation $q = a$ and the graphs $(q, f_1(q))$ for $q < a$ and $(q, f_3(q))$ for $q > a$. Instead, when $\sigma c > \delta$ the phase space consists in six regions (see Fig.1b), separated by the vertical straight lines of equation $q = a^*$ and $q = a$, besides the graphs $(q, f_1(q))$ for $q < a$ and $(q, f_3(q))$ for $q > a$. In the regions R_1, R_3, R_4 and R_6 the definition of the map is as in Fig.1a; in Region R_2 it is defined as $(q_{t+1}, k_{t+1}) = (f_1(q_t), g_2(k_t))$; in the regions R_5 and R_6 of Fig.1b the definition of the map is almost the same. They differ only in the contraction factor of the function $g_2(k_t) = (1 - \xi)k_t$ which is defined via two different constants (as given in (3)): in the region R_6 it is $\xi = \delta$, while in the region R_5 it is $\xi = \sigma c$.

We remark that in the rightmost regions, R_1, R_5 and R_6 the definition of the map is a contraction in both variables, so that the dynamics of $T(q_t, k_t)$ are naturally forced to enter in other regions. However, this does not preserve the dynamics to be always bounded in the phase space, as points having a divergent trajectory may exist (in particular parameter settings).

The definition of the map in region R_4, given by $(q_{t+1}, k_{t+1}) = (f_2(k_t), g_1(q_t, k_t))$ $= (k_t, (1+\sigma(a-c-q_t))k_t)$ can be written also as a delayed logistic in the variable capital:

$$k_{t+1} = (1 + \sigma(a - c))k_t - \sigma k_t k_{t-1} \tag{11}$$

whose dynamics have been investigated by several authors (see e.g. Aronson et al. 1982). So it is well known that it has a unique fixed point different from $k = 0$, given by

$$k^* = (a - c) \tag{12}$$

and thus the two-dimensional map T has also a positive fixed point $(q, k) = (k^*, k^*)$ iff this point belongs to the region R_4.

Proposition. *The system defined in (7) and (8) has a unique fixed point different from* $(0, 0)$*, given by* (k^*, k^*)*, belonging to Region* R_4.

To prove the proposition we have to show that the fixed point belongs to that region and that no other fixed point exists in the other regions. That $k^* = (a-c)$ is always smaller that $\min(a^*, a)$ follows immediately. Then we have to prove that $k^* < f_1(k^*)$. And in fact we have $f_1(k^*) = k^*(1 + \mu(a - \nu(1 + i) - k^*)) = k^*(1 + \mu(c - \nu(1 + i)) = k^*(1 + m(\delta + i))$ which clearly is larger than k^*. Now we have to consider the definitions in the other regions. As remarked above, no fixed point can exist in the regions R_1, R_5 and R_6 where there are contractions (whose dynamics converge towards $(0, 0)$ outside these regions). The function defined in the region R_3, $(q_{t+1}, k_{t+1}) = (f_1(k_t), g_1(q_t, k_t))$, has no fixed point, as can immediately be seen, while the function defined in the region R_2, $(q_{t+1}, k_{t+1}) = (f_1(k_t), g_2(k_t))$ has the point $(q, k) = (a - \nu(1 + i), 0)$ which does not belong to the region R_2 and thus it is not a fixed point of our system.□

Regarding the local stability of the fixed point (k^*, k^*), the Jacobian matrix of the system defined in region R_4 is given by

$$J(q, k) = \begin{bmatrix} 0 & 1 \\ -k\sigma & 1 + \sigma(a - c) - q\sigma \end{bmatrix} \tag{13}$$

so that

$$J(k^*, k^*) = \begin{bmatrix} 0 & 1 \\ -\sigma(a - c) & 1 \end{bmatrix} \tag{14}$$

whose characteristic polynomial is given by $\mathcal{P}(\lambda) = \lambda^2 - \lambda + \sigma(a-c)$. The related eigenvalues are given by

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1 - 4\sigma(a-c)}}{2} \tag{15}$$

and are complex conjugated for $4\sigma(a-c) > 1$. The fixed point can loose stability via a Neimark Sacker bifurcation, occurring when the complex eigenvalues become equal to 1 in modulus, which occurs for $\sigma(a-c) = 1$, that is when

$$NS: \ \sigma(a - \nu(1+i) - m(\delta + i)) = 1. \tag{16}$$

For $\sigma(a-c) < 1$ the fixed point is locally attracting, while for $\sigma(a-c) > 1$ it is a repelling focus. As locally (in the region R_4) the map is smooth, the effect of this supercritical bifurcation is well known. Close to the bifurcation value, a locally attracting closed invariant curve Γ exists, surrounding the unstable fixed point, and the dynamics on Γ are either quasiperiodic (when the rotation number of the trajectory is irrational), or the closed curve Γ consists in a saddle-node connection of a pair of cycles of some period n (when the rotation number is rational) and the attracting set is the attracting cycle. It is obvious that for some set of parameter values this attracting closed curve Γ is completely included in the region R_4. However, a change occurs when a contact between this invariant set and the border of the region R_4 takes place. This is an example of border collision bifurcation. And what occurs to the system after this bifurcation is, in general, not predictable. However, in the continuous case, a collision of an invariant set different from a periodic orbit is often persistent after the crossing of the border. This is due to continuity, although, clearly, how much the persistence can survive depends on the definition of the new map involved after the crossing.

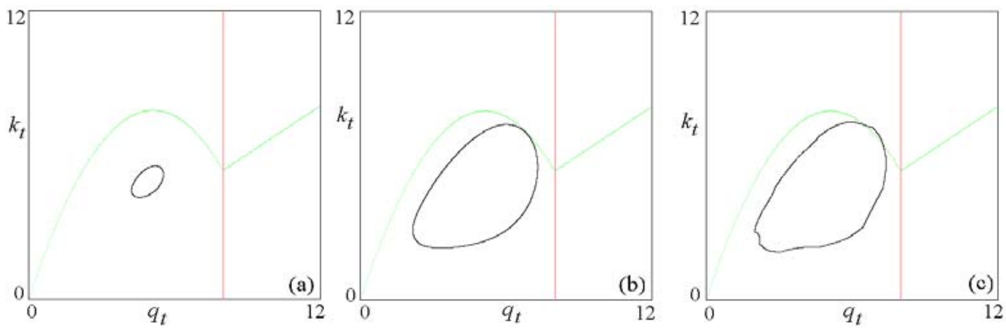

Fig.2 Closed invariant curve at $\delta = 0.9$. In (a) $\sigma = 0.205$; in (b) $\sigma = 0.22$; and in (c) $\sigma = 0.223$.

An example is shown in Fig.2 at the following parameter values:

$$a = 8, \ \nu = 1, \ m = 2, \ i = 0.1, \ \mu = 0.3 \tag{17}$$

which will be kept fixed in all our simulations in this work.

In Fig.2 the phase space is as shown in Fig.1a. In Fig.2a, soon after the Neimark Sacker bifurcation, the closed curve is quite far from the boundary of regions R_4 and R_5. In Fig.2b, increasing the value of σ, the closed curve is close to the boundary of region R_4, and the boundary is crossed in Fig.2c. Here the effect is not so dangerous, as in fact the closed curve persists, even if now the dynamics on it depend on two different definitions, and the invariant set crosses the two regions R_3 and R_4 (causing qualitative changes in the shape of Γ). In the next section we shall see that border collision bifurcations of cycles may lead to more dangerous dynamic effects.

3 Border collision bifurcations of cycles.

In the following we analyze the global properties of the dynamics in greater detail; we are particularly interested in the role of borders and constraints that were introduced with a genuine economic rational. Since, as remarked in the previous section, it is difficult to predict the global behavior of the map after a border collision bifurcation of a cycle (which, as explained below, always refers to a pair of cycles), also our investigations depend strongly on numerical simulations. Throughout the section, we concentrate on a variation in the parameters (σ, δ), which are our central parameters since δ measures the downward rigidity of capacity, and σ relates to cautious and aggressive entrepreneurial behavior, and a low (high) value of σ relates to cautious (aggressive) behavior. We keep the other parameter fixed as in (17). The result, in our case, is that we can distinguish between changes in the dynamics due to usual behaviors occurring as in smooth maps[1], and changes due to the constraints which have been introduced with an economic meaning.

To begin with, we note that the regions in the phase space represent different economic regimes: While regions R_1 and R_6 involve a zero price, regions R_2, R_3, R_4 and R_5 exhibit positive prices and each of it represents a specific regime within a boom and bust cycle: In regions R_2 and R_3 the intended quantity adjustment is not constrained by the existing capacity, or put differently, those regimes are characterized by excess capacity. In region R_3, positive or negative capacity adjustments evolve unbounded, whereas in region R_2 (given the high quantity and thus the low price) downward capacity adjustment is bounded by the depreciation rate – entrepreneurs would like to reduce capacity quicker than the depreciation rate allows. In regions R_4 and R_5 entrepreneurs would like to increase quantity above the existing capacity – however, quantity adjustment is constraint by the existing capacity; in region R_4 this is combined with an unconstrained (positive or negative) capacity adjustment whereas in region R_5 the intended (comparatively strong) reduction in capacity is restricted by the deprecation rate.

Fig.2 has already shown that the dynamics after the Neimark Sacker bifurcation of the fixed point – that is in Region R_4 – sooner or later enters also Region

[1] not only locally, but also globally, as contact bifurcations and homoclinic bifurcations of cycles.

R_3. In order to analyze more systematically the regions which are visited by the attracting set after the Neimark Sacker bifurcation of the fixed point, we present a numerical simulation in the two-dimensional parameter plane (σ, δ) while keeping the other parameter fixed as in (17). We remark, however, that the resulting picture takes into account only the attracting set reached with an initial condition in the phase space (q, k) quite close to the fixed point (k^*, k^*), and does not capture coexisting attractors, which are not so rare in our model.

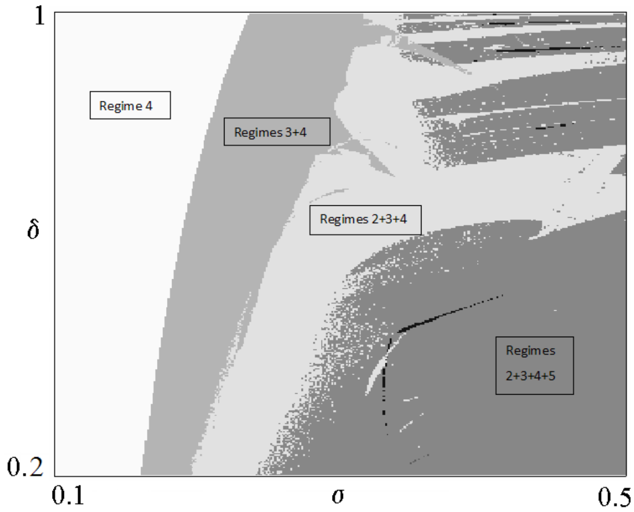

Fig.3 The regions denote the different regimes involved in the asymptotic state of the model. In the black points the system involves regime 3+4+5.

Fig.3 summarizes the results and shows how the nature of the attracting set vary with a variation in the parameters (σ, δ). For each parameter combination the grey shade indicates the regimes the dynamic path enters over the time. Region R_4, in which output adjustment is constrained by existing capacity, is clearly involved for all parameter combinations. Quite plausibly, regions R_2 and R_5, in which the downward rigidity of capacity adjustment is binding, is involved on the attracting set only for higher values of σ (i.e. with more aggressive entrepreneurial capacity adjustments) and/or lower values of δ, i.e. more severe downward rigidities in capacity adjustment. The broad dynamic pattern revealed in Fig.3 thus in line with the possible regimes switches involved in boom and bust cycles. The complex structure, however, is surprising.

Fig.4 completes the analysis of the global dynamics in the same parameter plane by showing for each (σ, δ) combination the period of the appertaining attracting cycle. Each color corresponds to a specific periodicity and a few periods are explicitly indicated in Fig.4. The white points of Fig.4 represent an attracting set as a closed invariant curve, or a chaotic attractor, or a cycle of period greater than 45. The region with "1" corresponds to the stability region of the fixed point. Its boundary is given by the NS curve (whose equation is in

(16)). Close to that curve there exist a strip of white points, associated with the existence of a closed attracting curve Γ. The black curve denoted with "r" denotes the kind of map. By using the definition of the cost parameter in (6) we have that the inequality $\sigma c \leq \delta$ holds iff $\delta \geq \frac{\sigma(\nu(1+i)+i)}{1-\delta m}$.

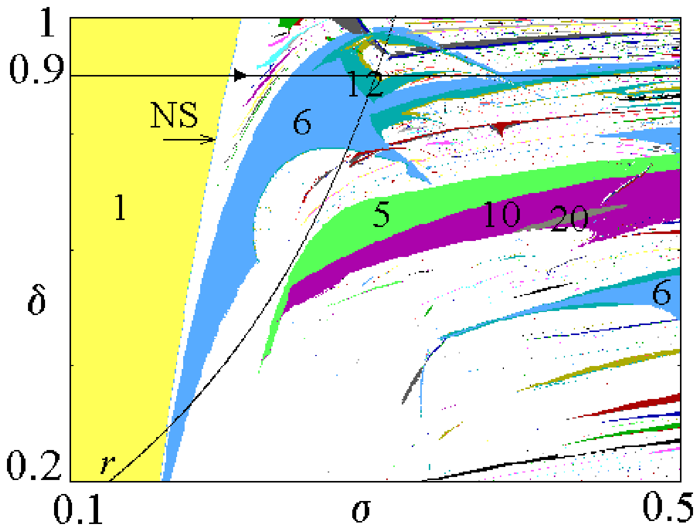

Fig.4 Two-dimensional parameter plane (σ, δ) at fixed $a = 8$, $\nu = 1$, $m = 2$, $i = 0.1$, $\mu = 0.3$. The different colors denotes regions associated with cycles of different periods. The curve denoted NS represents the Neimark-Sacker bifurcation of the fixed point.

Thus the curve

$$(r): \quad \delta = \frac{\sigma(\nu(1+i)+i)}{1-\delta m}$$

separates the two possible phase spaces: for parameters taken above it the system is defined via four regions, as shown in Fig.1a, while for parameters taken below it the system is defined via six regions, as shown in Fig.1b. The horizontal line at $\delta = 0.9$ shown in Fig.4 corresponds to a one-dimensional bifurcation diagram, in which the asymptotic states of q_t are reported as a function of the only parameter σ, as shown in Fig.5.

In order to summarize note that the global dynamics – as shown in Fig.3, 4 and 5 – is characterized by many abrupt and drastic changes. We will argue in the following using the bifurcation diagramme in Fig.5 that many of those changes actually involve border collision bifurcations and we will illustrate their possible effects on the dynamics path.

At the point A ($\sigma = 0.22$) of Fig.5 we have the contact of the curve Γ with the upper boundary of the region R_4 already commented in the previous section, in Fig.2b. A border collision bifurcation occurs that changes the shape of the attractor without drastically modifying its nature. The bifurcation occurring at point B ($\sigma = 0.245$) does not involve any border – it is a smooth saddle-node bifurcation at which the invariant curve disappears and which leads to a pair of 6-cycles, an attracting node and a saddle. As σ is increased, the transition to chaos follows the standard period doubling route. However, the transition to other periodic orbits is often associated with border collision bifurcations, i.e. with the collision of an invariant set with one of the borders of the regions. As an example, the transition occurring at the point C ($\sigma = 0.34193$) illustrates this type of bifurcation.

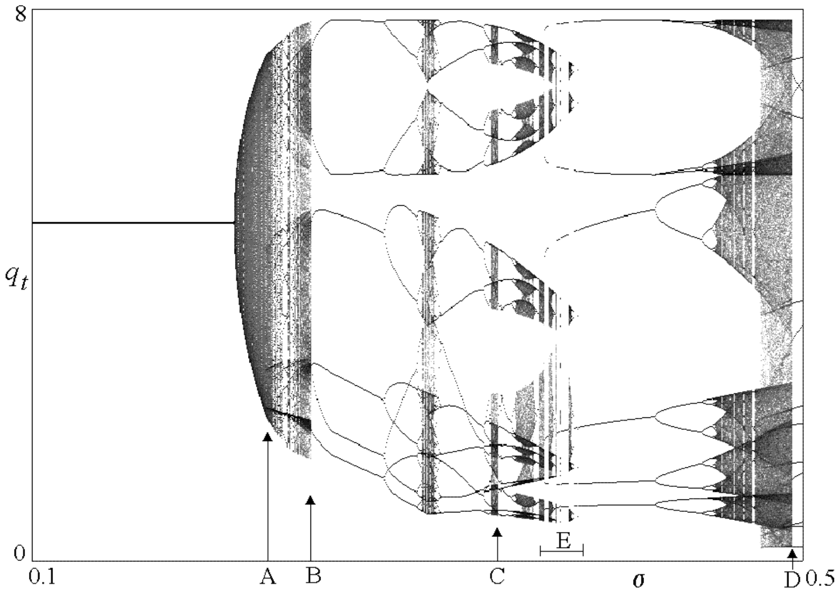

Fig.5 One-dimensional bifurcation diagram of q_t as a function of σ at $\delta = 0.9$. The points A, B, C, D correspond to $\sigma = 0.22$, $\sigma = 0.245$, $\sigma = 0.34193$, $\sigma = 0.4946$, respectively. The interval E is enlarged in Fig.7a.

In Fig.6a it is shown the attracting set, constituting in six chaotic pieces, and suddenly, at $\sigma = 0.34194$ an attracting cycle (of period 18) appears (see Fig.6b), which persist, attracting, for an interval of values in σ. This transition is associated with a border collision saddle-node bifurcation. Let us briefly comment the bifurcation decreasing the parameter.

When the 18-cycle exists, it is associated with a companion saddle 18-cycle, and two periodic points one of the node and one of the saddle are approaching and merging on a border, here the line $q = a^*$ (in Fig.6b we can see that one periodic point is almost on the border), after which the pair of cycles disappear. The attracting set thus changes suddenly, here leading to a 6-pieces chaotic attractor, shown in Fig.6a. Comparing the bifurcation occurring at point B (smooth saddle-node bifurcation) with the one at point C (border collision saddle-node bifurcation) we remark that the dynamic effects are very similar. However, the smooth saddle-node bifurcation is associated with a standard bifurcation and can be locally detected via the eigenvalues of the cycles. That is, at the smooth saddle-node bifurcation one of the eigenvalue is equal to +1. Thus, the occurrence of a smooth saddle-node bifurcation can be found using econometric methods (the time series exhibits a unit root) but it has no obvious economic rational. In contrast, at a border collision saddle-node bifurcation the eigenvalues of the cycles are not approaching the value +1 and its occurrence can no longer be predicted via the eigenvalues of the cycles. In that sense, its occurrence is more "dangerous", more unexpected. However, the role played by the eigenvalue in smooth system is now replaced by the borders of the regions. When a periodic point is close to a border that defines some region, a "collision" with the border is expected to occur, after which it is difficult to predict what will be the attracting set. It is important to note that the borders – and thus the collisions with it – have a genuine economic rational.

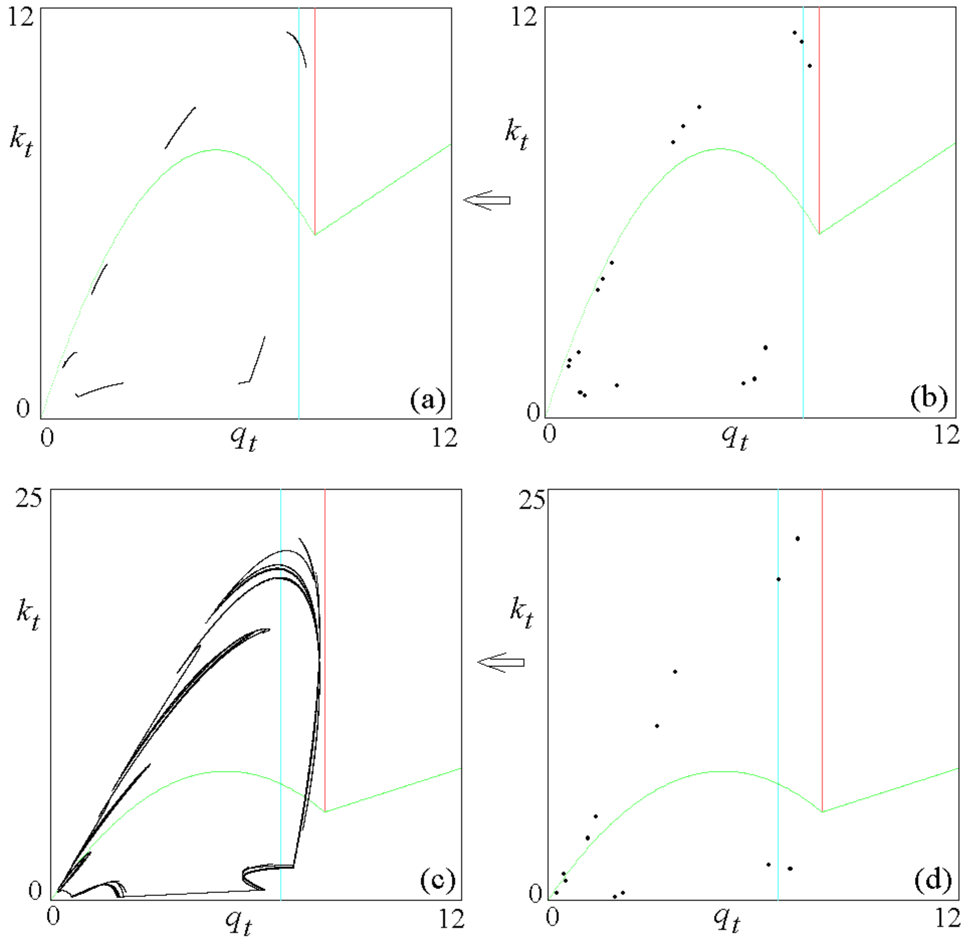

Fig.6 Attractor at the parameters as given in (17) and $\delta = 0.9$ along the horizontal path shown in Fig.4. In (a) at $\sigma = 0.34193$ effect of the border collision bifurcation occurring in (b) at $\sigma = 0.34194$. In (c) at $\sigma = 0.4945$ effect of the border collision bifurcation occurring in (d) at $\sigma = 0.4946$.

Another example of a border collision saddle-node bifurcation is found in Fig.5, at point D: Decreasing σ from 0.5 the attracting set initially is a cycle of period 13 (with a co-existing period 13 saddle), which suddenly disappear at the point σ. In Fig.6d an attracting 13-cycle is shown to exist, with periodic points in the 4 regions R_2, R_3, R_4 and R_5, and we can see that a periodic point is almost on the border of the line $q = a^*$ (separating the regions R_2 and R_3). Even if its eigenvalues are not close to $+1$ a border collision is expected to occur – one of the points of the attracting period-13 cycle (an one point of the coexisting period-13 saddle) collides exactly with the border, after which the two cycles disappear. In Fig.6c, at $\sigma = 0.4945$, we can see the drastic change in the dynamics, as the resulting attractor is a one-piece chaotic set, crossing the 4 regions R_2, R_3, R_4 and R_5.

The border collision saddle-node bifurcations seen so far involved a sudden change in the nature of the attracting set. However, it may also involve the sudden (dis)appearance of a pair of cycles (saddle and node). This behavior is related with the property of multistability, which often occurs in piecewise smooth systems. As an example, let us consider the segment denoted by E in Fig.5 and enlarged in Fig.7a. In Fig.7a we can see that in the interval between the point A ($\sigma = 0.366$) and B ($\sigma = 0.3853$) there are two coexisting attractors, one drawn in black and one in red (a 6-cycle). Let us analyze the behavior starting at $\sigma = 0.39$ where only a stable 6-cycle exists (in red in Fig.7a) which attracts almost all the points in the phase space. Decreasing the parameter σ, at $\sigma = 0.3853$ (point B in Fig.7a) another stable 6-cycle appears, marked in black in Fig.7a, in pair with a saddle 6-cycle. Fig.7c shows both the attracting cycles in the point B, the red one existing also before and after and the newly born black one. Note that the black cycle has a periodic point very close to the boundary between the regions R_3 and R_4. It follows that the black cycle is born via a border collision saddle-node bifurcation. While before the bifurcation (i.e. for σ higher than the value in point B ($\sigma = 0.3853$)) almost all the points in the phase space are attracted by the 6-cycle in red, after the appearance of the other (black) attractor the points of the phase space are shared between the two attractors. In Fig.7d we illustrate the shape of the two basins for the two attractors in Fig.7c. n Fig. 7d points in red (black) indicate initial conditions that are attracted to the red (black) cycle of period 6 shown in Fig 7c. We can see that the basins have a complex (fractal) structure. This is due to the existence, in the phase space, of a chaotic repeller: a Cantor set of points with infinitely many unstable cycles with homoclinic orbits, which create a complex, unobservable, repelling net, which however belongs to the frontier of the two basins (and it is visible in Fig.7d as the set of points separating the regions with the two colors).

In Fig.7a, as the parameter σ is decreased from the point B, we can see a period doubling route to chaos of the 6-cycle in black, always coexisting with the red 6-cycle. The red 6-cycle (dis)appears as the parameter σ crosses the point A. Before the disappearance there is coexistence between a 6-piece chaotic attractor and the red 6-cycle, as shown in Fig.7b, where we can also see that one point of the red 6-cycle is very close to the border separating the regions R_3 and

R_4 (and the related basins have also a complex structure). Therefore, at point A a stable 6-cycle (in red) appears/disappear via a border collision saddle-node bifurcation, in pair with a saddle 6-cycle. For smaller values of σ only the 6-pieces chaotic attractor is left. Considering the same bifurcation as the parameter σ is increased, at $\sigma = 0.365$ a 6-pieces chaotic attractor exists, at the point A a stable 6-cycle appears via a border collision saddle-node bifurcation, marked in red in Fig.7a, and from the wider Fig.5 we can see that also this red 6-cycle undergoes a period doubling route to chaos.

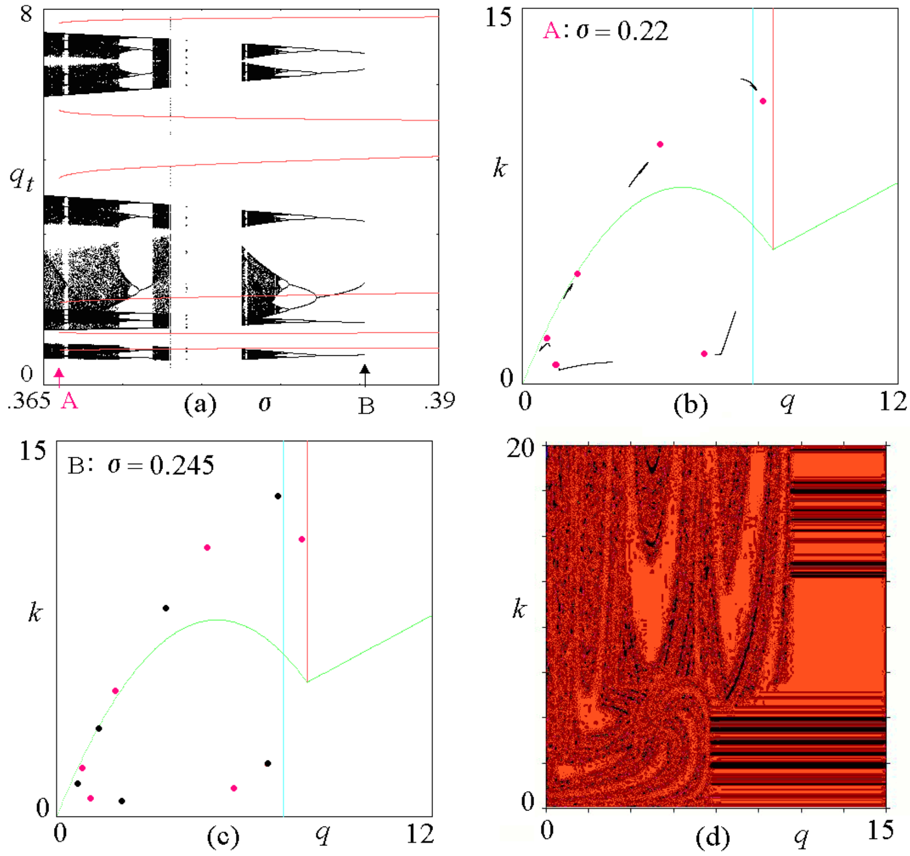

Fig.7 In (a) enlarged part of the bifurcation diagram shown in Fig.5 at $\delta = 0.9$. In (b) the two coexistent attractors existing when the parameters are in the point A ($\sigma = 0.366$). In (c) the two coexistent attractors existing when the parameters are in the point B ($\sigma = 0.3853$). In (d) the basins of attraction of the two different attractors shown in (c) at $\sigma = 0.3853$.

Note that in the parameter range associated with two coexisting attractors the related basins of attraction have always a complex structure. This is due to the fact that such border collision bifurcations occur after the destruction of the closed invariant curve associated with the Neimark Sacker bifurcation of the fixed point, and its destruction is often associated with chaotic dynamics or chaotic repellers. The complexity of the basins of attraction leads to a strong sensitivity of the dynamics to the initial conditions and/or parameters. In fact, a change in the position of the point in the phase space, due for example to a perturbation of some parameter or of the initial condition as a consequence of some shock, may lead to the transition towards a different attracting set. For example a cyclical path associated with a periodic orbit (the 6-cycle) may be lead to a chaotic state, or *vice versa*.

4 Conclusions

Boom and bust cycles are widely documented in the literature on industry dynamics and are one famous example for cyclical behavior of economic variables. Rigidities and delays in capacity adjustment in combination with bounded rational behavior have been identified as central driving forces. In this paper, we have constructed a model that features only these two elements and we have shown that this is indeed sufficient to produce a boom and bust dynamics. The bifurcation diagrams summarizing the dynamic behavior reveal complex cycles and in particular also abrupt changes in the nature of these cycles. Being on purpose oversimplified, the model admittedly looses many a realistic feature. However, its simplicity allowed us to apply new insights from the mathematical theory of piecewise smooth dynamic systems, in particular results from the theory of border collision bifurcations. The definition of our map is indeed only piecewise smooth – i.e. its definition differs in distinct regions of the phase space. Moreover the different regions in the phase space are not uniquely determined, as these can be 4 or 6 depending on the values of the economic parameters. This complicated structure in the definition of the map is not an arbitrary assumption made for analytic convenience or for generating "interesting" results; instead, it is the very existence of delays and rigidities in capacity adjustment – one of the core elements of our analysis – that introduces boundaries into the definition of the dynamic map. One of the boundaries separating the phase space follows from the fact that capacity cannot adjusted immediately and that – as a consequence – current output is constraint by existing capacity; another boundary reflects the fact that capacity reductions cannot exceed depreciation. In addition, we took into account non-negativity constraints for prices and quantities. The very existence of those – economically motivated – boundaries introduces the possibility of border collision bifurcation – occurring when one point of an attracting set collides with a border and giving rise to abrupt changes in the nature of the attractor or giving even rise to entirely new attractors. We showed in the paper various examples of border collision bifurcation illustrating how the dynamics may be affected by such a bifurcation. Our main point – that goes well beyond the scope of the paper – is that the very existence of borders such as capacity constraints or nonnegativity constraints may lie behind abrupt changes in the dynamic behavior of economic variables and gives them a genuine economic rational.

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