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On a Lemma of Schachermayer

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Abstract

In this paper we prove a topological lemma on real valued random variables which implies the basic ingredients for the proof of the Fundamental Theorem of Asset Pricing in the two period case. In particular, previous results of Stricker, [9], and of Schachermayer, [6], are special cases of our result. Our proof is considerably shorter and more transparent than previous proofs of related special cases.

Let $L_0(\Omega, \mathcal{A}, P_0)$ be the space of real valued random variables endowed with the topology of convergence in P_0-measure and let $\mathcal{B}$ be a sub-σ-field. The main result is concerned with $\mathcal{B}$-modules, i.e. subspaces which are modules for the ring of $\mathcal{B}$-measurable random variables. Let $C \subseteq L_0$ be a cone which is closed under multiplication with nonnegative $\mathcal{B}$-measurable functions. Let K and L be $\mathcal{B}$-modules and assume that K is finitely generated. Our main result asserts: If $L - C$ is closed and if $(K + L) \cap C = \{0\}$ then $K + L - C$ is closed, too.

1 Introduction

Let $(\Omega, \mathcal{A}, P_0)$ be a probability space and let $L_0(P_0)$ be the space of all (P_0-equivalence classes of) real-valued random variables endowed with the topology of convergence in P_0-measure. Let $\mathcal{B}$ be a sub-σ-field of $\mathcal{A}$. Consider random variables $X_1, X_2, \ldots, X_d$ in $L_0(P_0)$ and define

$$K := \left\{ h = \sum_{k=1}^{d} \theta_k X_k : \ \theta_k \in L_0(\Omega, \mathcal{B}, P_0) \right\}$$

The so-called Fundamental Theorem of Asset Pricing in its simplest two-period form states the the equivalence of the following two assertions.

(NA) **No Arbitrage**: $K \cap L_0^+(P_0) = \{0\}$.

(MM) **Existence of equivalent „martingale" measures**: *There is a probability measure $Q \sim P_0$ with bounded P_0-density such that $X_k \in L_1(Q)$ and $E_Q(X_k|\mathcal{B}) = 0$ for $k = 1, 2, \ldots, d$.*

It is well-known that the Fundamental Theorem of Asset Pricing for finitely many periods is an easy consequence of the equivalence $(NA) \Leftrightarrow (MM)$. The implication $(MM) \Rightarrow (NA)$ is trivial. The reverse implication $(NA) \Rightarrow (MM)$ is a more delicate matter. A proof for finite σ-fields has been given for the first time by Harrison and Kreps, 1979, [4]. The first proof of the general case is due to Dalang, Morton and Willinger, 1990, [1]. Since that time several attempts have been made to simplify the proof of this basic fact. Remarkable examples are the papers by Schachermayer, 1992, [6], and by Kabanov and Kramkov, 1994, [5]. It should be noted that the theorem in full generality (continuous time, infinite time interval) has been proved by Delbaen and Schachermayer, 1994, [2], and 1997, [3].

Let us reconsider the two-period case. There are two basic assertions concerning topological properties of K.

[1] *AMS 1991 subject classifications:* 60G99, 46A99, 62B20.

[2] *Key words and phrases:* Asset pricing, no-arbitrage, equivalent martingale measures, closed convex cones.

(1.1) LEMMA *(Stricker, [9], Lemma 2)*
The set K is closed in $L_0(P_0)$.

(1.2) LEMMA *(Schachermayer, [6], Lemma 2.1)*
Condition (NA) implies that $K - L_0^+(P_0)$ is closed in $L_0(P_0)$.

It is worth noting that without an assumption like (NA) the cone $K - L_0^+(P_0)$ need not be closed in $L_0(P_0)$. This follows from an example given in [6].

In the present paper we will prove an assertion which covers Lemmas (1.1) and (1.2) as special cases. The proof of this assertion only relies on basic measure theory and one single Hahn-Banach argument. Moreover, our proof is considerably shorter than those of Stricker and Schachermayer.

For the statement of our main result we require two definitions.

(1.3) DEFINITION *Let us call a subspace $L \subseteq L_0(P_0)$ a $\mathcal{B}$-modul if $\beta f \in L$ for every $f \in L$ and every $\mathcal{B}$-measurable function β.*

(1.4) DEFINITION *Let us call a cone $C \subseteq L_0(P_0)$ a $\mathcal{B}$-cone if $\beta f \in C$ for every $f \in C$ and every $\mathcal{B}$-measurable Funktion $\beta \geq 0$.*

Our main result is the following lemma.

(1.5) LEMMA *Let L be a $\mathcal{B}$-modul and C a $\mathcal{B}$-cone. If $L - C$ is closed in $L_0(P_0)$ and if $(K+L) \cap C = \{0\}$ then $K + L - C$ is closed in $L_0(P_0)$.*

For $L = C = \{0\}$, Lemma (1.5) is Stricker's Lemma (1.1). If we put $L = \{0\}$ and $C = L_0^+(P_0)$ then we obtain Schachermayer's Lemma (1.2). We will prove Lemma (1.5) for $d = 1$. Then the validity of the assertion for each finite $d \in \mathbb{N}$ follows by induction.

It is interesting to note that for the trivial σ-field $\mathcal{B} = \{\emptyset, \Omega\}$ the assertion of Lemma (1.5) is a special case of an easy fact on topological vector spaces.

(1.6) LEMMA *Suppose that E is a topological vector space. Let $L \subseteq E$ be a subspace and let $C \subseteq E$ be a cone such that $L - C$ is sequentially closed in E. If $K \subseteq E$ is a finite dimensional subspace such that $(K + L) \cap C = \{0\}$ then $K + L - C$ is sequentially closed in E.*

Proof: It is sufficient to prove the assertion for a one-dimensional subspace K. Thus, let $K = \text{span}\{x\}$. If $x \in L - C$ or $-x \in L - C$ then $(K + L) \cap C = \{0\}$ implies that $x \in L$ and the assertion is obvious. So we may assume that both $x \notin L - C$ and $-x \notin L - C$.

Let $z_n = \lambda_n x + y_n \to z$ where $(y_n) \subseteq L - C$. The proof is finished if we show that $\limsup_n |\lambda_n| < \infty$. Assuming $\limsup_n \lambda_n = \infty$ we can take a subsequence such that $x + y_n/\lambda_n \to 0$. This implies $-x \in \overline{L - C} = L - C$ which is not possible. The case $\liminf_n \lambda_n = -\infty$ is handled similarly. □

The preceding lemma remains true (with identical proof) if we replace „sequentially closed" by „closed".

The proof of Lemma (1.5) is given in section 2. It is a straightforward extension of the proof of Lemma (1.6) to the case of $\mathcal{B}$-modules and $\mathcal{B}$-cones.

In the remaining part of this section we will discuss another attempt of proving the Fundamental Theorem for the two period case. A recent and remarkable paper is by Kabanov and Kramkov, [5]. Let us briefly describe their results.

First, we observe that there is always a probability measure $P \sim P_0$ with bounded P_0-density such that $X_k \in L_1(P)$, $k = 1, 2, \dots, d$. Therefore we may assume w.l.g. that $X_k \in L_1(P_0)$ for $k = 1, 2, \dots, d$. If $M \subseteq L_1(P_0)$, we denote by $\overline{M}$ the closed hull of M in $L_1(P_0)$.

The following assertion is the main result of Kabanov and Kramkov.

(1.7) THEOREM *(Kabanov and Kramkov, [5], Theorem 3)*
Assume that $X_k \in L_1(P_0)$, $k \in 1, 2, \ldots, d$ *and denote* $\widetilde{K} := K \cap L_1(P_0)$. *Then the assertions* (NA), (MM) *and*

$$(KK): \quad \overline{(\widetilde{K} - L_1^+(P_0))} \cap L_1^+(P_0) = \{0\}$$

are equivalent.

Kabanov and Kramkov prove that $(NA) \Rightarrow (KK)$ and $(KK) \Rightarrow (MM)$.

It should be noted that $(NA) \Rightarrow (KK)$ is an immediate consequence of Schachermayer's Lemma (1.2). Nevertheless, Kabanov and Kramkov do not make use of Lemma (1.2), but give an independent direct proof of $(NA) \Rightarrow (KK)$. The assertion of Lemma (1.2) is not proved by Kabanov and Kramkov. It will turn out that our proof of Lemma (1.5) is also considerably shorter and more transparent than the proof of $(NA) \Rightarrow (KK)$ by Kabanov and Kramkov.

The proof of the second implication $(KK) \Rightarrow (MM)$ can be simplified by the application of an exhaustion argument or a theorem by Halmos and Savage (cf. Schachermayer, 1994, [7], and Kabanov and Kramkov, 1994, [5]), each of which reduces the proof to a simple Hahn-Banach separation. Since this is exactly the separation argument we shall apply in our inductive proof of Lemma (1.5) it is isolated in section 3 for the reader's convenience.

For completeness and in view of its elegance let us summarize the proof of the Fundamental Theorem of Asset Pricing (two period case), based on Schachermayer's Lemma and the the final argument of Kabanov and Kramkov.

(1.8) THEOREM *Assertion* (NA) *implies assertion* (MM).

Proof: Let $P_1 \sim P_0$ be a probability measure with bounded P_0-density such that $X_k \in L_1(P_1)$, $k = 1, 2, \ldots, d$. By Lemma (1.2) the set $K - L_0^+(P_0)$ is closed in $L_0(P_1)$. By (NA) the set $K - L_0^+(P_0)$ does not contain indicators of sets with positive P_1-measure. Hence, by Lemma (3.1) applied for $L = K$ and $C = L_0^+(P_0)$, for every $A \in \mathcal{A}$ with $P_1(A) > 0$ there exists $Z_A \in L_\infty^+(P_1)$ such that

$$\int_A Z_A \, dP_1 > 0, \; E_{P_1}(Z_A X_k | \mathcal{B}) = 0, \; k = 1, 2, \ldots d.$$

Define $Q_A := Z_A P_1 / \int Z_A \, dP_1$. By a Lemma of Halmos and Savage, which is familiar in statistics (see e.g. [8], Lemma 20.3) there is a probability measure $Q \in \overline{\text{co}}\{Q_A : A \in \mathcal{A}\}$ such that $Q \sim P_1 \sim P_0$. It is easy to see that the probability measure Q can be chosen with a bounded P_1-density. □

Thus, in order to complete the proof of Theorem (1.8) it remains to prove Lemma (1.5) which gives Schachermayer's Lemma (1.2). A simple proof of this lemma is the subject of the next section.

2 The proof of the main result

Let $X \in L_0(P_0)$ and let $K = \{h = \theta X : \theta \in L_0(\Omega, \mathcal{B}, P_0)\}$. We will prove the following assertion.

(2.1) PROPOSITION *Let* L *be a* $\mathcal{B}$*-modul and* C *a* $\mathcal{B}$*-cone such that* $L - C$ *is closed in* $L_0(P_0)$. *If*

$$(K + L) \cap C = \{0\} \tag{2.1}$$

then $K + L - C$ *is closed in* $L_0(P_0)$.

Lemma (1.5) follows from Proposition (2.1) by induction. It should be noted that our proof runs completely parallel to the proof of Lemma (1.6). A first step is the following reduction lemma.

(2.3) LEMMA *Suppose that the assumptions of Proposition (2.1) are satisfied. Then there is a set $B_0 \in \mathcal{B}$ such that*

(1) $1_{B_0}X \in L$, *and*

(2) $P(B \setminus B_0) > 0$ *implies* $1_B X \notin L - C$ *and* $-1_B X \notin L - C$.

Proof: Let $\mathcal{S} = \{B \in \mathcal{B} : 1_B X \in L - C\}$. From (2.2) it follows that $1_B X \in L$ for all $B \in \mathcal{S}$. Moreover, it is easy to see, that $\mathcal{S}$ is hereditary and closed under finite unions.

Let $s := \sup\{P_0(B) : B \in \mathcal{S}\}$. Since $L - C$ is closed there exists some $B_0 \in \mathcal{S}$ with $P_0(B_0) = s$. The maximality of s implies immediately that for $B \in \mathcal{B}$

$$1_B X \in L - C \;\Rightarrow\; P(B \setminus B_0) = 0. \tag{2.2}$$

On the other hand, if $-1_B X \in L - C$ for some $B \in \mathcal{B}$, then by (2.2) we obtain $-1_B X \in L$. Hence, $1_B X \in L$ which again implies $P(B \setminus B_0) = 0$. □

Part (1) of the preceding lemma implies that for every $h \in K + L - C$ there is a $\mathcal{B}$-measurable function θ such that

$$h = \theta X + f \text{ with } f \in L - C, \text{ and } \theta = 0 \text{ on } B_0.$$

After these preparations we are in a position to give the proof of Proposition (2.1).

Proof: (of Proposition (2.1))

Let $(h_n) \subseteq K + L - C$ such that $h_n \overset{P_0}{\to} h$ and $h \in L_0(P_0)$. Choosing a subsequence we may achieve that $h_n \to h$ P_0-a.e. Let

$$h_n = \theta_n X + f_n \quad \text{with } f_n \in L - C, \text{ and } \theta_n = 0 \text{ on } B_0. \tag{2.3}$$

We will prove that $P_0\{\limsup_n |\theta_n| = \infty\} = 0$. If this is done then taking convex combinations (cf. Schachermayer, [6], Lemma 3.2) we may achieve that $\theta_n \to \theta$ P_0-a.e. This implies that $h - \theta X$ is in the $L_0(P_0)$-hull of $L - C$, hence by assumption even in $L - C$. Thus, we arrive at $h \in K + L - C$.

For the rest of the proof we replace P_0 by a probability measure $P_1 \sim P_0$ with bounded P_0-density such that $\sup_n |h_n| \in L_1(P_1)$. This is possible since

$$\limsup_n |h_n| = |h| < \infty \; P_0\text{-a.e.} \;\Rightarrow\; \sup_n |h_n| < \infty \; P_0\text{-a.e.}.$$

It follows that $h \in L_1(P_1)$ and $||h_n - h||_{P_1} \to 0$. Moreover, we may choose P_1 such that $X \in L_1(P_1)$. Then the functions f_n satisfy $E_{P_1}(|f_n|\,|\mathcal{B}) < \infty$ P_1-a.e.

Let $B_1 = \{\limsup_n \theta_n = \infty\}$. Assume that $P(B_1) > 0$. Since $B_1 \subseteq B_0'$, Lemma (2.3) implies $-1_{B_1}X \notin L - C$. By Corollary (3.2) there exists $Z \in L_\infty(P_1)$ such that $E_{P_1}(Zf_n|\mathcal{B}) \le 0$ for each $n \in \mathbb{N}$ and

$$\int_{B_1} ZX \, dP_1 < 0. \tag{2.4}$$

Now (2.5) implies

$$\begin{aligned} -\infty < E_{P_1}(Zh|\mathcal{B}) &\le \liminf_n E_{P_1}(Zh_n|\mathcal{B}) \\ &\le \liminf_n \Big(\theta_n E_{P_1}(ZX|\mathcal{B})\Big) \end{aligned} \tag{2.5}$$

and hence $B_1 \subseteq \{E_{P_1}(ZX|\mathcal{B}) \ge 0\}$, contradicting (2.6).

Let $B_2 = \{\liminf_n \theta_n = -\infty\}$. Assume that $P(B_2) > 0$. Since $B_2 \subseteq B_0'$, Lemma (2.3) implies $1_{B_2}X \notin L - C$. By Lemma (3.1) there exists $Z \in L_\infty(P_1)$ such that $E_{P_1}(Zf_n|\mathcal{B}) \le 0$ for each $n \in \mathbb{N}$ and

$$\int_{B_2} ZX \, dP_1 > 0. \tag{2.6}$$

Again (2.5) implies (2.7) and hence $B_2 \subseteq \{E_{P_1}(ZX|\mathcal{B}) \le 0\}$, in contradiction to (2.8). □

3 The basic separation lemma

(3.1) LEMMA *Let L be a $\mathcal{B}$-modul and C a $\mathcal{B}$-cone and suppose that $L - C$ is closed in $L_0(P_0)$. If $Y \in L_1(P_0)$, $Y \notin L - C$, then there exists $Z \in L_\infty(P_0)$ such that $E_{P_0}(ZY) > 0$ and $E_{P_0}(Zf|\mathcal{B}) \leq 0$ whenever $f \in (L - C) \cap L_1(P_0)$.*

Proof: The set $(L - C) \cap L_1(P_0)$ is closed in $L_1(P_0)$. Since $Y \notin L - C$ there is $Z \in L_\infty(P_0)$ such that

$$\int Zh\, dP_0 \leq c < \int ZY\, dP_0 \quad \text{whenever } h \in (L - C) \cap L_1(P_0).$$

Since $0 \in L - C$ we have $c \geq 0$.
Let $f \in (L - C) \cap L_1(P_0)$ and $B := \{E_{P_0}(Zf|\mathcal{B}) > 0\}$. Since $\lambda 1_B f \in (L - C) \cap L_1(P_0)$ for every $\lambda > 0$ it follows that

$$\int_B Zf\, dP_0 \leq 0 \text{ whence } P_0(B) = 0.$$

□

For the proof of Theorem (1.8) the preceding lemma is applied to $L = K$ and $C = L_0^+(P_0)$. In this case the assertion implies $Z \geq 0$ and $E_{P_0}(X_k|\mathcal{B}) = 0$ if $X_k \in L_1(P_0)$.
For the proof of Lemma (1.5) we need a little more.

(3.2) COROLLARY *The assertion of Lemma (3.1) is even satisfied for all $f \in L - C$ for which $E_{P_0}(|f|\,|\mathcal{B}) < \infty$ P_0-a.e.*

Proof: For $n \in \mathbb{N}$ let $B_n := \{E_{P_0}(|f|\,|\mathcal{B}) \leq n\}$. Then $f_n := 1_{B_n} f \in (L - C) \cap L_1(P_0)$ and by Lemma (3.1) we have $E_{P_0}(f_n|\mathcal{B}) \leq 0$ for each $n \in \mathbb{N}$. Now the assertion follows from

$$|E_{P_0}(f|\mathcal{B}) - E_{P_0}(f_n|\mathcal{B})| \leq E_{P_0}(|f|(1 - 1_{B_n})|\mathcal{B}) \stackrel{P_0}{\to} 0.$$

□

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