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Paper

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M/M/1 queues under N-policy involving batches

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Abstract

In this paper the transient solutions of $M^R/M/1$ and $M/M^R/1$ models are derived in discrete and continuous time.

1 Introduction

In this paper we will consider two types of queueing process:

1. Customers arrive according to a Poisson process with rate λ and are served in batches of fixed size $R \geq 1$. The service time distribution is exponential with mean $1/\mu$. The system opens with the arrival of the N−th ($N \geq R$) customer and continues service until there are less than R customers waiting. Then the system closes and remains closed until a new queue of length N has built up.

2. Customers arrive in batches of fixed size $R \geq 1$ according to a Poisson process with rate λ. The service times are exponentially distributed with mean $1/\mu$.

The system opens with the arrival of the $N-$th customer and continues service until there are no customers left. Then the system closes and remains closed until a new queue of length N has built up.

2 Queues with batch service

Let $Q(t)$ be the number of customers waiting at time t and let $Q(0)=m\geq 0$. Then the following relations hold for a time interval of infinitesimal length Δ:

$$\begin{aligned}
\mathrm{P}(Q(t+\Delta)=k|Q(t)=k-1) &= \lambda\Delta+o(\Delta) \qquad (1)\\
\mathrm{P}(Q(t+\Delta)=k-R|Q(t)=k) &= \mu\Delta+o(\Delta)\\
&\quad \text{if the system is open at time } t\\
\mathrm{P}(Q(t+\Delta)=k|Q(t)=k) &= 1-(\lambda+\mu)\Delta+o(\Delta)\\
&\quad \text{if the system is open at time } t\\
\mathrm{P}(Q(t+\Delta)=k|Q(t)=k) &= 1-\lambda\Delta+o(\Delta)\\
&\quad \text{if the system is closed at time } t
\end{aligned}$$

All other probabilities being of order $o(\Delta)$.

Furthermore define $A(t)=$ the number of arrivals during all completed idle periods. If the system is idle initially, we agree that the arrivals during this period do not contribute to $A(t)$. This should hold even if the system starts empty. Arrivals during such a period will be treated separately.

It will be convenient to discretize the model in the following way: segment the time interval $(0,t)$ into n slots of length $\Delta=t/n$ and define the discrete time analogue of the process $Q(t)$ as $Q_n, n\geq 0$ with the following convention:

$$\mathrm{P}(Q_i=k)=\mathrm{P}(Q(\tau)=k,(i-1)\Delta<\tau\leq i\Delta) \qquad \text{for} \qquad 0\leq i\leq n$$

Similary define A_n as the discrete time analogue of $A(t)$. To simplify the notation set $\lambda\Delta=\alpha, \mu\Delta=\gamma$ and $1-(\lambda+\mu)\Delta=\beta$. For a detailed discussion of this discretization procedure including various limit theorems see Böhm W. and Mohanty S.G. (1990).

2.1 The model in discrete time

With the process Q_n we associate a lattice path in the plane, the associated path (AP). It is constructed as follows: a horizontal step represents an arrival, a vertical step a batch departure. There are, however, two types of diagonal steps corresponding to the event that Q_n does not change, which may happen with probability β during busy periods and with probability $1 - \alpha$ during an idle period.

A typical path may look like Figure 1:

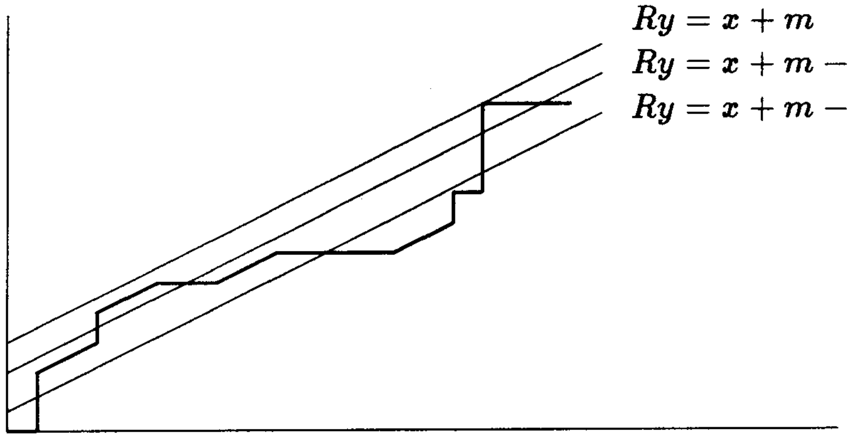

Figure 1

The description of such paths is facilitated by some basics results which are summarized in the following lemma:

Lemma 1 *1.) Let $p_n(a)$ =P(an unrestricted path of length n terminates on the line $Ry = x + a$), then*

$$p_n(a) = \sum_{i\geq 0} \binom{i(R+1)-a}{i}_+ \binom{n}{i(R+1)-a} \alpha^{Ri-a}\gamma^i\beta^{n+a-i(R+1)} \tag{2}$$

where $(.)_+$ means that this binomial coefficient is equal to 0 whenever the upper parameter is < 0.

2.) Let $q_n(a)$ =P(a path of length n does not touch or cross the line $Ry = x$ and terminates on the line $Ry = x - a, a > 0$), then

$$q_n(a) = \sum_{i\geq 0} \frac{a}{a+i(R+1)} \binom{a+i(R+1)}{i} \binom{n-1}{a+i(R+1)-1} \alpha^{a+Ri}\gamma^i\beta^{n-a-i(R+1)} \tag{3}$$

and $q_n(0) = \delta_{n0}$*, the Kronecker delta.*

3.) Let $s_n(a,k)$ *=P(a path of length n does not touch or cross the line* $Ry = x + a, a \geq 0$*, and terminates on the line* $Ry = x + k, k < a$*), then*

$$s_n(a,k) = p_n(k) - p_n(a) * q_n(a-k) \tag{4}$$

where "" denotes the convolution over n, in particular, for sequences* f_n *and* g_n

$$f_n * g_n = \sum_{i=0}^{n} f_i g_{n-i}$$

4.) Let $w_n(a,i)$ *=P(a path of length n does not cross the line* $Ry = x + a, a \geq 0$ *except at the end and terminates on the line* $Ry = x + a + R - i, 0 \leq i < R$*), then*

$$w_n(a,i) = \gamma s_{n-1}(a+1, a-i) \tag{5}$$

where $w_0(a,i) = 0$*. In the special case* $a = 0$ *we have*

$$w_n(0,i) = \tag{6}$$

$$= \sum_{j\geq 0} \frac{1+i}{j(R+1)+1+i} \binom{j(R+1)+1+i}{j} \binom{n-1}{i+j(R+1)} \alpha^{Rj+i} \gamma^{j+1} \beta^{n-1-i-j(R+1)}$$

Proof.

1.) There are $\binom{n_1+n_2}{n_1}$ paths consisting of n_1 horizontal and n_2 vertical steps. Every such path passes through $n_1 + n_2 + 1$ lattice points. Let us now place n_3 diagonal steps of β–type in those points. This can be done in $\binom{n_1+n_2+n_3}{n_3}$ different ways, hence

$$p_n(a) = \sum_{\substack{n_1+n_2+n_3=n \\ n_1-Rn_2=-a}} \binom{n_1+n_2}{n_1} \binom{n_1+n_2+n_3}{n_3} \alpha^{n_1} \gamma^{n_2} \beta^{n_3}$$

which after some simplification yields (2).

2.) By the generalized ballot theorem (see for instance Mohanty, p. 8, Theorem 3) the number of paths consisting of n_1 horizontal and n_2 vertical steps, which lie entirely below the line $Ry = x$ is given by

$$\frac{n_1 - Rn_2}{n_1 + n_2} \binom{n_1+n_2}{n_1}$$

Every such paths passes through $n_1 + n_2 + 1$ lattice points and in any of those points, except for the first one, we may insert $\beta-$type steps, hence

$$q_n(a) = \sum_{\substack{n_1+n_2+n_3=n \\ n_1-Rn_2=a}} \frac{n_1 - Rn_2}{n_1 + n_2} \binom{n_1+n_2}{n_1} \binom{n_1+n_2+n_3-1}{n_3} \alpha^{n_1} \gamma^{n_2} \beta^{n_3}$$

which is equivalent to (3).

3.) The set of all paths of length n, which terminate on the line $Ry = x + k$ decomposes into the set of paths with the required property and the set of paths which touch or cross the line $Ry = x+a$. Consider now the latter set in more detail: every path in this set may be split into two segments, an unrestricted segment which ends on the line $Ry = x + a$ and a segment which starts on $Ry = x + a$ and ends on $Ry = x + k$ without touching $Ry = x + a$ except at the beginning. Thus the result follows from (2) and (3).

4.) Any such path has to terminate on the line $Ry = x + a - i$ after $n - 1$ steps followed by a $\gamma-$type step. Therefore the result follows using (4). In the special case $a = 0$ we have to count paths to the point (Rn_2, n_2) which do not cross the line $Ry = x$. The number of those paths is given by Rohatgi(1972) and equals

$$\frac{1+i}{n_2(R+1)+1+i} \binom{n_2(R+1)+1+i}{n_2}$$

Hence

$$w_n(0,i) = \gamma \sum_{\substack{n_1+n_2+n_3=n-1 \\ n_1-Rn_2=i}} \frac{1+i}{n_2(R+1)+1+i} \binom{n_2(R+1)+1+i}{n_2} \binom{n-1}{n_3} \alpha^{n_1} \gamma^{n_2} \beta^{n_3}$$

■

Remark. Actually $q_n(a)$ and $w_n(a,i)$ are probabilitites of stopping times. This is immediately apparent for $w_n(a,i)$. That this is even true for $q_n(a)$ may be seen, if we look at the reverse path defining $q_n(a)$. This path starts on the line $Ry = x - a$ and ends with a first passage through the origin. This special feature of $q_n(a)$ and $w_n(a,i)$ will become important, when we study the limiting behaviour of the above defined quantities.

The case $A_n = 0$ is particulary simple, since this event implies that there is no completed uncensored idle period.

Let ${}_{BB}\mathrm{P}(Q_n = k, A_n = 0|Q_0 = m)$ denote the probability, that at time n there are k customers in the system, the server is busy and there is no completed idle period, conditional on the event that initially there have been m customers waiting and the server has been busy at time 0. Furthermore let ${}_{IB}\mathrm{P}(Q_n = k, A_n = 0|Q_0 = m)$ be defined similary with the difference that the server was idle initially. Analogously ${}_{BI}\mathrm{P}()$ and ${}_{II}\mathrm{P}()$ mean that the server is busy (idle) initially and idle at time n.

By means of Lemma 1 we may formulate the following theorem:

Theorem 1 *Let* $g_n(a) = \binom{n-1}{a-1}\alpha^a(1-\alpha)^{n-a}$ *and* $h_n(a) = \binom{n}{a}\alpha^a(1-\alpha)^{n-a}$, *then for* $m, k \geq R$*:*

$$ {}_{BB}P(Q_n = k, A_n = 0|Q_0 = m) = s_n(m - R + 1, m - k) \tag{7} $$

for $0 \leq m < N, k \geq R$*:*

$$ {}_{IB}P(Q_n = k, A_n = 0|Q_0 = m) = g_n(N - m) * s_n(N - R + 1, N - k) \tag{8} $$

for $m \geq R, 0 \leq k < N$*:*

$$ {}_{BI}P(Q_n = k, A_n = 0|Q_0 = m) = \sum_{0 \leq i \leq \min(k, R-1)} w_n(m - R, i) * h_n(k - i) \tag{9} $$

for $0 \leq m < N, 0 \leq k < N$*:*

$$ \begin{aligned} &{}_{II}P(Q_n = k, A_n = 0, Q_i \geq N \text{ for some } i, 0 < i < n|Q_0 = m) = \\ &= \sum_{0 \leq i \leq \min(k, R-1)} w_n(N - R, i) * h_n(k + N - m - i) \end{aligned} \tag{10} $$

and for $0 \leq m < N$ *and* $0 \leq k < N$*:*

$$ {}_{II}P(Q_n = k, A_n = 0, Q_i < N, 0 \leq i \leq n|Q_0 = m) = h_n(k - m) \tag{11} $$

Proof. To prove (7) note that the AP is a lattice path which does not cross the line $Ry = x + m - R$ and terminates on the line $Ry = x + m - k$, since at time n there are k customers waiting. The result is established using formula (4) of Lemma 1.

To deal with the second case we split the AP into two segments, a path starting with N customers, which is completely described by (7) and a segment consisting of α- and $(1-\alpha)$-type steps, which has to end with an α-type step, thus giving rise to a negative binomial distribution.

To prove (9) observe that an idle period may start with $0 \leq i < R$ customers. In any of these cases the AP has to cross the line $Ry = x + m - R$ and terminate on the line $Ry = x + m - i$ after $0 \leq l \leq n$ steps. In the remaining $n - l$ steps there have to be $k - i$ arrivals, where $i \leq \min(k, R-1)$. An application of (5) yields the result.

The arguments for proving (8) and (9) carry over directly to the case (10).

Formula (11) is self evident. ■

The case $A_n > 0$ is a bit more complicated. However, this implies that there is at least one completed and uncensored idle period. Therfore it will be convenient to split the AP into two segments: the first segment represents Q_n up to the end of the last idle period. This segment terminates on the line $Ry = x + m - N$. The second segment corresponds to the last not necessarily completed busy period. Its behaviour is completely described by Theorem 1. Let us disregard diagonal steps for the moment and assume that the system is initially busy with $m \geq R$ customers waiting. Then the enumeration of segments of the first type is accomplished by the following lemma:

Lemma 2 *The number of paths consisting of n_1 horizontal steps corresponding to arrivals during busy periods, a horizontal steps due to arrivals during idle periods and n_2 batch departures equals the number of paths from $(0,0)$ to (n_1, n_2), terminating with a first passage on the line $Ry = x + m + a - N$.*

Proof. Define for an arbitrary path $\mathcal{P}$ and a point $P = (x_0, y_0) \in \mathcal{P}$ the height of p as $h = Ry_0 - x_0$. Consider now the paths in Figure 2 and Figure 3.

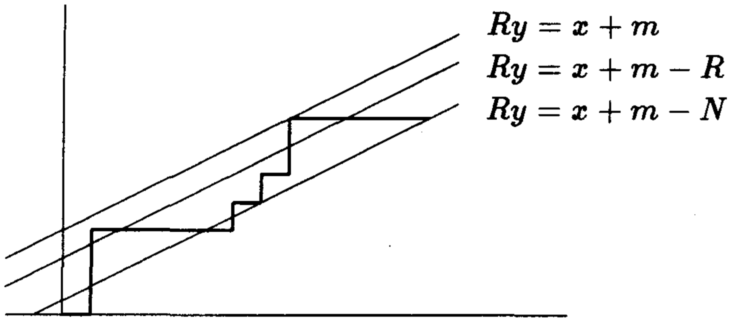

Figure 2

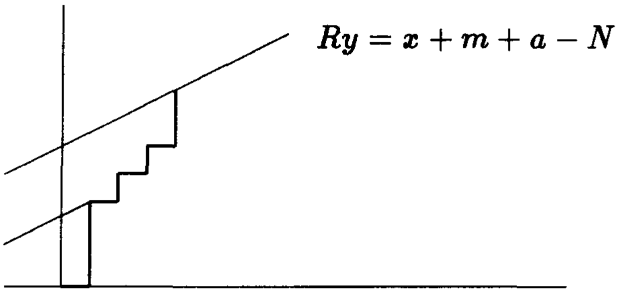

Figure 3

Note that idle periods are represented by horizontal path segments, which start between the line $Ry = x + m$ and (but not on) line $Ry = x + m - R$ and terminate on the line $Ry = x + m - N$.

Suppose we have ν idle periods with individual lengths $c_1, c_2, \ldots, c_\nu$ and $\sum c_i = a$. The points marking the beginnings of the idle periods have heights $h_i = m - N + c_i$. Now cut out successively the horizontal steps corresponding to the ν idle periods. Then h_1 will not change. Let h_i^* be the heights after the transformation. Hence $h_1^* = h_1$. But h_2 changes, in particular $h_2^* = h_2 + c_1$, since we have cut out c_1 horizontal units. Now continue and cut out c_2 horizontal steps corresponding to the second idle period. h_2^* will not change, but we find that $h_3^* = h_3 + c_1 + c_2$ and in general

$$h_i^* = h_i + c_1 + c_2 + \ldots + c_{i-1}$$

Thus

$$h_\nu^* = h_\nu + \sum_{i=1}^{\nu-1} c_i$$

$$
\begin{aligned}
&= m - N + \sum_{i=1}^{\nu} c_i \\
&= m - N + a
\end{aligned}
$$

Consider now the reverse mapping; define $h_0 = m - R$ and look for the next ladder index with height $h_1 > h_0$. In this point insert $h_1 - h_0 + N - R$ horizontal steps, thus giving rise to a new path. On this path look for the next ladder index with height $h_2 > h_1$ and insert $h_2 - h_1 + N - R$ horizontal units. Continue this procedure up to the last ladder index. Doing this, the original path is reconstructed in a unique way. ■

Let us now reintroduce diagonal steps in the same manner as in Lemma 1 and define $b_n(a)$ =P(a path of length n ends with a first passage on the line $Ry = x + a, a > 0$). By Lemma 1 we have

$$
\begin{aligned}
b_n(a) &= w_n(a-1, R-1) \quad &\text{if} \quad a > 0 \\
&= \delta_{0n} \quad &\text{if} \quad a = 0
\end{aligned}
$$

In our queueing context $b_n(a)$ has an interesting interpretation: starting with a busy system with initially $a \geq R$ customers in the queue, $b_n(a)$ equals the probability that the system becomes empty (not only idle) for the first time after n steps.

Putting our results together we may now prove the main theorem:

Theorem 2 *For $m, k \geq R, a > 0$*

$$
{}_{BB}P(Q_n = k, A_n = a | Q_0 = m) = b_n(m + a - N) * g_n(a) * s_n(N - R + 1, N - k) \quad (12)
$$

for $k \geq R, 0 \leq m < N, a > 0$

$$
{}_{IB}P(Q_n = k, A_n = a | Q_0 = m) = b_n(a) * g_n(a + N - m) * s_n(N - R + 1) \quad (13)
$$

for $0 \leq k < N, 0 \leq m < N, a > 0$

$$
\begin{aligned}
&{}_{II}P(Q_n = k, A_n = a | Q_0 = m) = \\
&= b_n(a) * \sum_{0 \leq i \leq \max(k, R-1)} w_n(N - R, i) * h_n(k - i + a + N - m) \quad (14)
\end{aligned}
$$

and for $m \geq R, 0 \leq k < N, a > 0$

$$
\begin{aligned}
&{}_{BI}P(Q_n = k, A_n = a | Q_0 = m) = \\
&= b_n(m + a - N) * \sum_{0 \leq i \leq \max(k, R-1)} w_n(N - R, i) * h_n(a + k - i) \quad (15)
\end{aligned}
$$

Proof. For $m, k \geq R, a > 0$ we may split the AP into two segements, exactly as it is done in the derivation of Lemma 2. The first segment has probability $b_n(m+a-N)$. Then there are a arrivals during completed idle periods, this subprocess ending with an arrival. The associated probability is $g_n(a)$. Together with (7) the result follows.

To prove (13) we note, that we could equally well start with N customers, where we now have $N - m$ arrivals to complete the first left censored idle period and a arrivals during the other idle periods, the sequence of these events ending with an arrival with probability $g_n(a + N - m)$. Splitting the path as above, the first segment has probability $b_n(a)$. Using (7) we get (13).

In the case $0 \leq k < N, 0 \leq m < N$ it will be convenient to consider first the arrivals during all idle periods, completed and uncompleted. The first leftcensored idle period needs $N - m$ arrivals for its completion. Then there are a arrivals during completed and uncensored idle periods. The last right censored idle period starts with i customers, where $0 \leq i \leq \min(k, R - 1)$. Since at time n there are k customers waiting, we require additionally $k - i$ arrivals. Looking at this subprocess of arrivals as a whole, we find, that it is a Bernoulli sequence with success = arrival, the sequence not necessarily terminating with a success. Hence the associated probability is $h_n(k - i + a + N - m)$. Let us now look at the rest of the path. The first busy period starts with N customers. The process continues up to the end of the last completed idle period. Since we have already cut out the horizontal steps corresponding to arrival during completed idle periods, this segment has probability $b_n(a)$ by Lemma 2. The last busy period starts with N customers and together with (9) formula (14) follows.

The proof of (15) is essentially the same as that of (14). ■

2.2 Continuous time results

We recall that we have set $\alpha = \lambda\Delta, \gamma = \mu\Delta$ and $\beta = 1 - \alpha - \gamma$, where $\Delta = t/n$. Now let $n \to \infty$. Then the follwing lemma will give the basic results:

Lemma 3 *1.) Let* $\lim_{n\to\infty} p_n(a) = p_t(a)$, *then*

$$p_t(a) = e^{-(\lambda+\mu)t} \rho^{-\frac{a}{R+1}} I_{-a}^{R}(zt) \tag{16}$$

where $z = 2\lambda^{\frac{R}{1+R}}\mu^{\frac{1}{1+R}}$ *and* $I^R_{-a}(zt)$ *denotes the Luchak function of order* $-a$ *(Luchak 1956).*

2.) Let $\lim_{n\to\infty} q_n(a) = q_t(a)\,dt$*, then*

$$q_t(a) = \frac{a}{t} e^{-(\lambda+\mu)t} \rho^{\frac{a}{R+1}} I^R_a(zt) \tag{17}$$

where z *is defined as above. Furthermore define* $q_t(0) = \delta(t)$*, Dirac's deltafunction.*

3.) Let $\lim_{n\to\infty} s_n(a,k) = s_t(a,k)$*, then*

$$s_t(a,k) = p_t(k) - p_t(a) * q_t(a-k) \tag{18}$$

where "$*$" denotes the convolution operator, in particular

$$f_t * g_t = \int_0^t f_s g_{t-s} ds$$

4.) Let $\lim_{n\to\infty} w_n(a,i) = w_t(a,i)\,dt$*, then*

$$w_t(a,i) = \mu s_t(a,i) \tag{19}$$

and in the special case $a = 0$*:*

$$w_t(0,i) = \frac{1+i}{t} e^{-(\lambda+\mu)t} \rho^{\frac{i-R}{R+1}} I^R_{1+i}(zt \tag{20}$$

5.) Let $\lim_{n\to\infty} b_n(a) = b_t(a)\,dt$*, then*

$$b_t(a) = w_t(a-1, R-1) \tag{21}$$

and $b_t(0) = \delta(t)$*.*

6.) Let $\lim_{n\to\infty} g_n(a) = g_t(a)\,dt$ *and* $\lim_{n\to\infty} h_n(a) = h_t(a)$*, then*

$$g_t(a) = \frac{\lambda t^{a-1} e^{-\lambda t}}{(a-1)!} \quad \text{and} \quad h_t(a) = \frac{(\lambda t)^a e^{-\lambda t}}{a!} \tag{22}$$

Proof.
1.) First note that $p_n(a)$ is actually the coefficient of v^{-a} in the function $(\alpha v + \beta + \gamma/v^R)^n$. This is by no means surprising, since $p_n(a)$ is the probability that a particle

where $z = 2\lambda^{\frac{R}{1+R}}\mu^{\frac{1}{1+R}}$ *and* $I^R_{-a}(zt)$ *denotes the Luchak function of order* $-a$ *(Luchak 1956).*

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6.) Let $\lim_{n\to\infty} g_n(a) = g_t(a)\,dt$ *and* $\lim_{n\to\infty} h_n(a) = h_t(a)$*, then*

$$g_t(a) = \frac{\lambda t^{a-1}e^{-\lambda t}}{(a-1)!} \qquad \text{and} \qquad h_t(a) = \frac{(\lambda t)^a e^{-\lambda t}}{a!} \tag{22}$$

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where z *is defined as above. Furthermore define* $q_t(0) = \delta(t)$*, Dirac's deltafunction.*

3.) Let $\lim_{n\to\infty} s_n(a,k) = s_t(a,k)$*, then*

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where "" denotes the convolution operator, in particular*

$$f_t * g_t = \int_0^t f_s g_{t-s} ds$$

4.) Let $\lim_{n\to\infty} w_n(a,i) = w_t(a,i)\,dt$*, then*

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and in the special case $a = 0$*:*

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5.) Let $\lim_{n\to\infty} b_n(a) = b_t(a)\,dt$*, then*

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$$g_t(a) = \frac{\lambda t^{a-1} e^{-\lambda t}}{(a-1)!} \qquad \text{and} \qquad h_t(a) = \frac{(\lambda t)^a e^{-\lambda t}}{a!} \tag{22}$$

Proof.

1.) First note that $p_n(a)$ is actually the coefficient of v^{-a} in the function $(\alpha v + \beta + \gamma/v^R)^n$. This is by no means surprising, since $p_n(a)$ is the probability that a particle

performing a one-dimensional random walk and starting in state 0 is in state a after n steps, where it moves one unit to the right with probability α, R units to the left with probability γ and remains at its current position with probability β. Hence

$$\begin{aligned}\lim_{n\to\infty} p_n(a) &= p_t(a) \\ &= \lim_{n\to\infty}[v^{-a}]\left(\frac{\lambda t}{n}v + 1 - \frac{(\lambda+\mu)t}{n} + \frac{\mu t}{nv^R}\right)^n \\ &= [v^{-a}]e^{-(\lambda+\mu)t}\exp\left(\lambda t v + \frac{\mu t}{v^R}\right)\end{aligned}$$

Now set $z = 2\lambda^{\frac{R}{1+R}}\mu^{\frac{1}{1+R}}$ and $\rho = \lambda/\mu$, then

$$p_t(a) = [v^{-a}]e^{-(\lambda+\mu)t}(zt/2)^{-a}\rho^{-\frac{a}{R+1}}\exp\left(v + \frac{(zt/2)^{R+1}}{v^R}\right) \tag{23}$$

From (23) it follows that the functions are well defined for all integer values of a. However a much stronger result can be established. The interested reader is referred to Srivastava and Kashyap (1982), p.42.

Extracting the coefficient of v^{-a} in(23) yields

$$p_t(a) = e^{-(\lambda+\mu)t}\rho^{-\frac{a}{1+R}}\sum_{i\geq 0}\frac{(zt/2)^{-a+i(R+1)}}{i!(i(R+1)-a)!}$$

with the convention that $1/(-k)! = 0$ for $k \in \mathbb{N}$.

2.) To prove (17) we recall the remark at the end of the proof of Lemma 1, namely that $q_n(a)$ is the probability function of a stopping variable. It has been shown by Böhm and Mohanty (1990, formula 9) that in this case

$$q_t(a) = \lim_{n\to\infty} q_n(a)\frac{n}{t}$$

Hence

$$\begin{aligned}q_t(a) &= \lim_{n\to\infty}\frac{n}{t}\sum_{i\geq 0}\frac{a}{a+i(R+1)}\binom{a+i(R+1)}{i}\binom{n-1}{a+i(R+1)-1}\times \\ &\quad \times\left(\frac{\lambda t}{n}\right)^{a+Ri}\left(\frac{\mu t}{n}\right)^i\left(1-\frac{(\lambda+\mu)t}{n}\right)^{n-a-i(R+1)} \\ &= \frac{a}{t}\lim_{n\to\infty} p_n(-a)\end{aligned}$$

3.) To prove (18) we need again a result due to Böhm and Mohanty (1990, formula 12), which shows that the discrete time convolution between a transition probability and the probability function of a stopping time converges to the continuous time convolution between the associated limiting functions.

4.) Here we are dealing again with the probability function of a stopping variable, therefore

$$\begin{aligned} w_t(a,i) &= \lim_{n\to\infty} w_n(a,i)\frac{n}{t} \\ &= \mu \lim_{n\to\infty} s_{n-1}(a+1,a-i) \end{aligned}$$

In the special case $a = 0$ we have

$$\begin{aligned} &w_t(0,i) = \\ &= \lim_{n\to\infty} w_n(0,i)\frac{n}{t} \\ &= \lim_{n\to\infty} \frac{i+1}{t} \sum_{j\geq 0} \frac{n}{j(R+1)+i+1} \binom{j(R+1)+i+1}{j} \binom{n-1}{i+j(R+1)} \times \\ &\quad \times \left(\frac{\lambda t}{n}\right)^{i+Rj} \left(\frac{\mu t}{n}\right)^{j+1} \left(1 - \frac{(\lambda+\mu)t}{n}\right)^{n-1-i-j(R+1)} \\ &= \rho^{-1}\frac{i+1}{t} \lim_{n\to\infty} p_n(-1-i) \end{aligned}$$

5.) This result follows directly from the definition of $b_n(a)$ and (19).

6.) selfevident ■

As an immediate consequence of Lemma 3 we have

Theorem 3 *For* $m, k \geq R$*:*

$$_{BB}P(Q(t) = k, A(t) = 0 | Q(0) = m) = s_t(m - R + 1, m - k) \tag{24}$$

for $0 \leq m < N, k \geq R$*:*

$$_{IB}P(Q(t) = k, A(t) = 0 | Q_0 = m) = g_t(N - m) * s_t(N - R + 1, N - k) \tag{25}$$

for $m \geq R, 0 \leq k < N$:

$$_{BI}P(Q(t) = k, A(t) = 0 | Q_0 = m) = \sum_{0 \leq i \leq \min(k, R-1)} w_t(m - R, i) * h_t(k - i) \quad (26)$$

for $0 \leq m < N, 0 \leq k < N$:

$$\begin{aligned} & _{II}P(Q(t) = k, A(t) = 0, Q_i \geq N \text{ for some } i, 0 < i < n | Q_0 = m) = \\ & = \sum_{0 \leq i \leq \min(k, R-1)} w_t(N - R, i) * h_t(k + N - m - i) \quad (27) \end{aligned}$$

and for $0 \leq m < N$ *and* $0 \leq k < N$:

$$_{II}P(Q(t) = k, A(t) = 0, Q_i < N, 0 \leq i \leq n | Q_0 = m) = h_t(k - m) \quad (28)$$

Theorem 4 *For* $m, k \geq R, a > 0$

$$_{BB}P(Q(t) = k, A(t) = a | Q_0 = m) = b_t(m + a - N) * g_t(a) * s_t(N - R + 1, N - k) \quad (29)$$

for $k \geq R, 0 \leq m < N, a > 0$

$$_{IB}P(Q(t) = k, A(t) = a | Q_0 = m) = b_t(a) * g_t(a + N - m) * s_t(N - R + 1) \quad (30)$$

for $0 \leq k < N, 0 \leq m < N, a > 0$

$$\begin{aligned} & _{II}P(Q(t) = k, A(t) = a | Q_0 = m) = \\ & = b_t(a) * \sum_{0 \leq i \leq \max(k, R-1)} w_t(N - R, i) * h_t(k - i + a + N - m) \quad (31) \end{aligned}$$

and for $m \geq R, 0 \leq k < N, a > 0$

$$\begin{aligned} & _{BI}P(Q(t) = k, A(t) = a | Q_0 = m) = \\ & = b_t(m + a - N) * \sum_{0 \leq i \leq \max(k, R-1)} w_t(N - R, i) * h_t(a + k - i) \quad (32) \end{aligned}$$

3 Queues with batch arrivals

Let us now assume that the customers arrive in batches of fixed size $R \geq 1$ according to a Poisson process with rate λ. Service times are exponentially distributed with mean $1/\mu$. The system opens with the arrival of the $N-$th customer. Since customers arrive in batches it is natural to set $N = \xi R, \xi \in \mathbf{N}$. Service continues until there are no customers left. Then the system closes and remains closed until a new queue of length N has built up.

The situation is now substantially simpler than it was the case of batch service. This is primarily due to the fact that now idle periods always start with zero customers. As in the previous section let $Q(t)$ be the number of customers waiting at time t, $Q(0) = m$ and define $A(t) =$ the number of arrivals during all completed idle periods. Arrivals during an initial left censored idle period will not contribute to $A(t)$. This convention holds even for a right censored idle period. Note that then always $A(t) = Ni$, where $i, i \geq 0$ is the number of completed idle periods.

We discretize the model in exactly the same way as in section 2 and define the discrete time analogues of $Q(t)$ and $A(t)$, viz. Q_n and A_n. With Q_n we may associate a lattice path in the plane which is now constructed as follows: a horizontal unit represents a departure, a vertical step a batch arrival. Again there are two types of diagonal steps corresponding to the event that Q_n does not change, which may happen during an idle and a busy period with probabilities $1-\alpha$ and β, respectively.

Disregarding diagonal steps and assuming that the system is busy initially with $m > 0$ customers, a typical path may look like figure 4.

Since A_n is a multiple of the number of completed idle periods it will be convenient to use the following notation:

Let ${}^{i}_{BB}\mathrm{P}(Q_n = k|Q_0 = m)$ =P(the system starting busy with $m > 0$ customers is busy for the $i-$ time (not counting the initial left censored busy period) at time n with k customers waiting), and ${}^{i}_{II}\mathrm{P}(Q_n = k|Q_0 = m)$ =P(the system starting idle with $m \geq 0$ customers is idle for the $i-$ time (not counting the initial left censored idle period) at time n with k customers waiting). Similary define the probabilities ${}^{i}_{BI}\mathrm{P}(Q_n = k|Q_0 = m)$ and ${}^{i}_{IB}\mathrm{P}(Q_n = k|Q_0 = m)$.

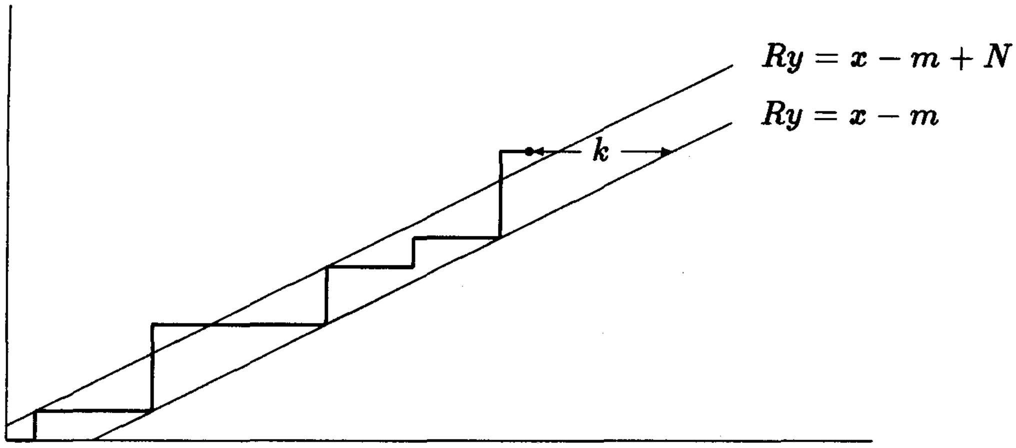

Figure 4

Theorem 5 *Let $\overline{s}_n(a,k), \overline{p}_n(a)$ and $\overline{q}_n(a)$ denote the probabilities defined in Lemma 1 with γ and α interchanged.Furthermore let $\kappa, \sigma \in \mathbf{N}$.*

1.)

$$ {}^{0}_{BB}P(Q_n = k | Q_0 = m) = \overline{s}_n(k, k-m) \tag{33}$$

2.) For $i > 0$ and $k \leq N$

$$ {}^{i}_{BB}P(Q_n = k | Q_0 = m) = \overline{s}_n(k, k-m-Ni) * g_n(\xi i) \tag{34}$$

and for $i > 0, k > N$:

$$ {}^{i}_{BB}P(Q_n = k | Q_0 = m) = [\overline{s}_n(k, k-m-Ni) - \overline{s}_n(k-N, k-m-Ni)] * g_n(\xi i) \tag{35}$$

3.) For $i \geq 0$

$$ {}^{i}_{II}P(Q_n = \sigma R | Q_0 = \kappa R) = \overline{q}_n(Ni) * h_n(\xi i + \sigma - \kappa) \tag{36}$$

4.) For $i \geq 1$

$$ {}^{i}_{IB}P(Q_n = k | Q_0 = \kappa R) = \overline{s}_n(k, k - N(i+1)) * g_n(\xi i - \kappa) \tag{37}$$

5.) For $i \geq 1$

$$ {}^{i}_{BI}P(Q_n = \sigma R | Q_0 = m) = \overline{q}_n(m + N(i-1)) * h_n(\xi(i-1) + \sigma) \tag{38}$$

Proof.

1. The AP starts at the origin and terminates after n steps on the line $Ry = x - m + k$ without touching the line $Ry = x - m$. Now look at the reversed path: it starts at the origin and terminates after n steps on the line $Ry = x - m + k$ without touching the line $Ry = x + k$. Hence using Lemma 1 (4) and interchanging the probabilities α and γ the result follows.

2. Consider the following mapping: cut out all vertical segements corresponding to arrivals during completed idle periods. The resulting path (see figure 5) has the following properties: it starts at the origin and terminates on the line $Ry = x - m - Ni$ but does not touch the line $Ry = x - m - N(i-1)$. This mapping is easily seen to be bijective. Now two cases arise, $k \le N$ and $k > N$. In the first case the path necessarily touches or crosses the line $Ry = x - m - N(i-1)$. The corresponding reversed path terminates after n steps on the line $Ry = x - m - Ni + k$ and does not touch the line $Ry = x + k$. Since there are ξi batch arrivals during the idle periods, this sequence ending with a batch arrival, formula (34) follows.

   In the case $k > N$ we use the following decomposition: the set of all paths terminating on the line $Ry = x - m - Ni + k$ without touching the line $Ry = x - m - Ni$ equals the union of the set of paths with the required property and the set of paths terminating on the line $Ry = x - m - Ni + k$ without touching the line $Ry = x - m - N(i-1)$. If we reverse the paths in the latter set we see that they terminate on the line $Ry = x - m - Ni + k$ and do not touch the line $Ry = x + k - N$. Using Lemma 1(4) and (34),(35) follows.

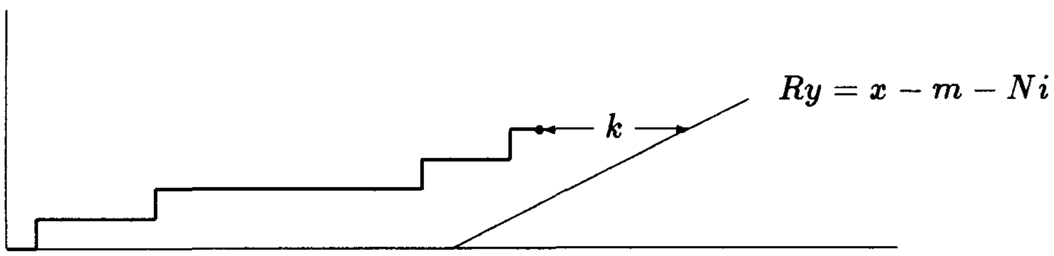

Figure 5

3. Consider first the left censored initial and the right censored idle period at time n. For the initial idle period to end $\xi - \kappa$ batch arrivals are required. During the last idle period there are σ batch arrivals. For the remaining part of the path we observe that the the system opens with N customers and has $i-1$ completed idle periods. The AP (after deleting the vertical steps due to arrivals during completed idle periods) terminates with a first passage on the line $Ry = x - Ni$. Thus the reversed path starts at the origin, terminates on the line $Ry = x - Ni$ and does not touch the line $Ry = x$ except at the beginning. The result follows using Lemma 1 (3).

4. The initial idle period requires $\xi - \kappa$ batch arrivals to terminate. The system starts then with N customers and has i completed idle periods. Therefore by (11) we get (37).

5. During the last right censored idle period there are σ batch arrivals and during the forgoing $i-1$ completed idle periods there have been $\xi(i-1)$. Consider now the path up to the beginning of the $i-$th idle period. Cutting out the vertical segments due to the completed idle periods we are left with a path terminating with a first passage on the line $Ry = x + m + N(i-1)$. Looking at the reversed path and using (36) the result follows. ■

The derivation of the corresponding continuous time result poses no new problems, Lemma 3 applies equally well to the "conjugate" probabilities $\overline{s}_n(a,k), \overline{q}_n(a)$ and $\overline{p}_n(a)$.

We may summarize these results in the following

Theorem 6 *Let $\overline{s}_t(a,k), \overline{p}_t(a)$ and $\overline{q}_t(a)$ denote the functions defined in Lemma 3 with λ and μ interchanged. Furthermore let $\kappa, \sigma \in \mathsf{N}$.*
1.)

$$_{BB}^{0}P(Q(t) = k|Q(0) = m) = \overline{s}_t(k, k-m) \tag{39}$$

2.) For $i > 0$ and $k \le N$

$$_{BB}^{i}P(Q(t) = k|Q(0) = m) = \overline{s}_t(k, k-m-Ni) * g_t(\xi i) \tag{40}$$

and for $i > 0, k > N$:

$$ {}^{i}_{BB}P(Q(t) = k|Q(0) = m) = [\bar{s}_t(k, k-m-Ni) - \bar{s}_t(k-N, k-m-Ni)] * g_t(\xi i) \quad (41) $$

3.) For $i \geq 0$

$$ {}^{i}_{II}P(Q(t) = \sigma R|Q(0) = \kappa R) = \bar{q}_t(Ni) * h_t(\xi i + \sigma - \kappa) \quad (42) $$

4.) For $i \geq 1$

$$ {}^{i}_{IB}P(Q(t) = k|Q(0) = \kappa R) = \bar{s}_t(k, k - N(i+1)) * g_t(\xi i - \kappa) \quad (43) $$

5.) For $i \geq 1$

$$ {}^{i}_{BI}P(Q(t) = \sigma R|Q(0) = m) = \bar{q}_t(m + N(i-1)) * h_t(\xi(i-1) + \sigma) \quad (44) $$

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