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Paper

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BEHAVIOUR OF THE LENGTH TEST FOR MEDIUM SAMPLE SIZES

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ABSTRACT

In this note it is shown that even for relatively large sample sizes the asymptotic distribution of the smoothed length as derived in Reschenhofer and Bomze (1991) should not be used for the determination of critical values. Therefore extended tables of critical values for both the 1% and 5% levels of significance generated by simulation are presented.

1. INTRODUCTION

A natural test statistic for the problem to decide whether or not a random sample with limits 0 and 1 comes from a uniform-$[0,1]$ distribution, is the Kolmogorov/Smirnov statistic which is given by the largest deviation of the empirical distribution function from the true distribution function. In the following, another test will be considered which has a clear geometrical interpretation too. Given that one wants to discriminate also against multimodal alternatives having a relatively small maximum deviation from the zero hypothesis, it might be advantageous to use the length of the graph of the c.d.f. (which is $\sqrt{2}$ under the zero hypothesis and exceeds this value otherwise) instead.

The following section presents the test statistic, which is obtained by replacing the theoretical by the empirical distribution function, and a smoothed version of it showing better asymptotic behaviour. Section 3 illustrates the need of care in using the asymptotic distribution derived before, and includes extended tables for both the 1% and 5% levels of significance.

2. TEST STATISTICS AND ASYMPTOTIC DISTRIBUTIONS

Consider an i.i.d. sample $X_1, \ldots, X_n$ with distribution function F, which throughout is assumed to be twice continuously differentiable with support $[0,1]$ and strictly positive derivative on $[0,1]$. Denote by

$$0 = X_{0:n} \leq X_{1:n} \leq \cdots \leq X_{n:n} \leq X_{n+1:n} = 1$$

the order statistics, and the spacings by

$$e_{kn} = X_{k+1:n} - X_{k:n}\,, \quad 0 \leq k \leq n\,.$$

Then the length of the linearly interpolated empirical distribution function based upon this sample is given by

$$\ell_n(F) = \sum_{k=0}^{n-1} \sqrt{\tfrac{1}{n^2} + e_{kn}^2} + e_{nn} = \tfrac{1}{n} \sum_{k=0}^{n-1} h(ne_{kn}) + e_{nn}\,,$$

where $h(y) = \sqrt{1+y^2}$. Equally well, one could also consider, instead of $\ell_n(F)$, the quantity

$$e_{0n} + \frac{1}{n} \sum_{k=1}^{n} h(ne_{kn})\,.$$

In the context of time series analysis, the analogous test for Gaussian white noise suggests the use of

$$L_n(F) = \frac{1}{n+1} \sum_{k=0}^{n} h\big((n+1)e_{kn}\big)\,,$$

where F is here the spectral distribution function, frequencies being measured as a fraction of π. Since $L_n(F)$ and $\ell_n(F)$ have the same asymptotic properties, we shall investigate $L_n(F)$ instead of $\ell_n(F)$, the main reason being the symmetry of the former quantity.

The length test as proposed in Reschenhofer and Bomze (1991) rejects the null hypothesis that $F = F_0$, the equidistribution function on $[0,1]$, if the length L_n is too large. The smoothed length test is obtained by replacement of the e_{kn} in L_n by the smoothed spacings

$$q_{kn} = \tfrac{1}{m} \textstyle\sum_{i=k-r}^{k+r} e_{in} = \tfrac{1}{m} t_{k-r:n}\,,$$

where $m = 2r+1 \sim \sqrt{n}$ and where $t_{j:n} = X_{j+m+1:n} - X_{j:n}$, $0 \leq j \leq n-m+1$, are the m-spacings of the sample $X_1, \ldots, X_n$. Hence the smoothed length test rejects the null hypothesis if

$$M_n(F) = \frac{1}{n+1} \sum_{k=r}^{n-r} h\big((n+1)q_{kn}\big)\,.$$

is too large.

The limiting distribution of $L_n(F_0)$ can be derived from results of Holst/Rao (1981) on the asymptotic behaviour of spacings. Similarly, one obtains the limiting distribution of $M_n(F_0)$ with the help of Hall's (1986) theory of higher order spacings. For details see Reschenhofer/Bomze (1991).

Theorem 1:

(a) Under the null hypothesis $F = F_0$,

$$\sqrt{n}[L_n(F_0) - \mu_{L,0}] \to \mathcal{N}(0, \sigma^2_{L,0}) \quad \text{in distribution as } n \to \infty$$

with

$$\mu_{L,0} = \mathrm{E}\,[h(Z)] = \int_0^\infty \sqrt{1+y^2}e^{-y}\,dy \approx 1.53886$$

and

$$\begin{aligned}\sigma^2_{L,0} &= \mathrm{Var}\,[h(Z)] - [\mathrm{Cov}\,(h(Z), Z)]^2 \\ &= 3 - \mu^2_{L,0} - [\int_0^\infty \frac{y^2}{\sqrt{1+y^2}}e^{y}\,dy]^2 \\ &\approx 3 - \mu^2_{L,0} - (0.78425)^2 \approx 0.0169\,.\end{aligned}$$

Here Z is a random variable with standard exponential distribution. The integrals were calculated by the IMSL-routine DQDAGI.

(b) Under the null hypothesis $F = F_0$,

$$n^{3/4}[M_n(F_0) - \mu_{M,0}] \to \mathcal{N}(0, \sigma^2_{M,0}) \quad \text{in distribution as } n \to \infty\,,$$

where

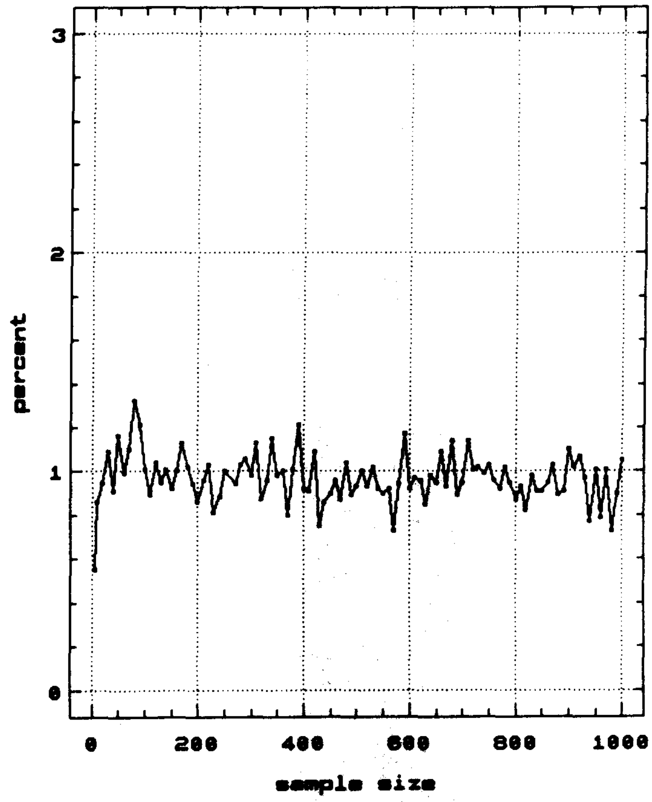
$$\mu_{M,0} = \sqrt{2} - \frac{7}{4\sqrt{2n}} \quad \text{and} \quad \sigma^2_{M,0} = \tfrac{3}{8}\,.$$

From Theorem 1 we see that $M_n(F_0) \to \sqrt{2}$ which is the true length of F_0. In contrast, we have $L_n(F_0) \to \mu_{L,0} > \sqrt{2}$ in probability as $n \to \infty$, i.e. the non-smoothed length test statistic asymptotically overestimates the true length. Furthermore note the difference in the convergence rates. Finally observe that $\mu_{M,0} < \sqrt{2}$ holds, the difference vanishing with rate $n^{-1/2}$ as $n \to \infty$. This difference compensates for boundary effects in the smoothing procedure.

3. SIMULATION RESULTS AND TABLES

In order to assess the suitability of the use of asymptotic distributions for the determination of critical values, a simulation study was carried out. 10,000 random samples were generated for $n+1 = 5, 10, 20, 30, \ldots, 1000$. Figures 1 and 2 show the portions of rejections at a nominal level of significance of 1% . For better readability, successive points are connected. The simulations suggest that convergence of the test statistics – although guaranteed by Theorem 1 – is rather slow. In particular, for the smoothed version the goodness of approximation strongly depends on the distance between $n+1$ and the next lower even square number. As r is taken to be the integer part of the square root of $\frac{n+1}{2}$, accuracy is extremly bad for $n+1 = 4k^2 - 1, k = 1, 2, \ldots$. To illustrate this fact sample sizes $4k^2 - 1$ and $4k^2, k = 1, 2, .., 15$ are additionally included in Figure 2 . At these sample sizes sharp peaks occur (cf. Figure 2). This phenomenon is due to the specific choice of the smoothing procedure which corresponds to the simplest possible, rectangular data window. Unfortunately, there are up to now no asymptotic results available for more sophisticated windows, for which one would expect less pronounced oscillations.

FIG. 1. Empirical sise of the length test for various sample sizes at a nominal level of significance at 1% ; determination of critical values was based on the asymptotic distribution.



Consequently, larger tables seem to be needed to apply the smoothed length test for medium sample sizes (see Table I). For $n + 1 \leq 100$, each value has been calculated by generating 100,000 pseudo-random sequences, and for larger sample sizes 30,000. Note that the differences in the values of Table I and those of Table I in Reschenhofer/Bomze (1991) are due to the fact that the spacings have been normalised after smoothing in Reschenhofer/Bomze (1991).

FIG. 2. Empirical size of the smoothed length test for various sample sizes at a nominal level of significance at 1% ; determination of critical values was based on the asymptotic distribution.

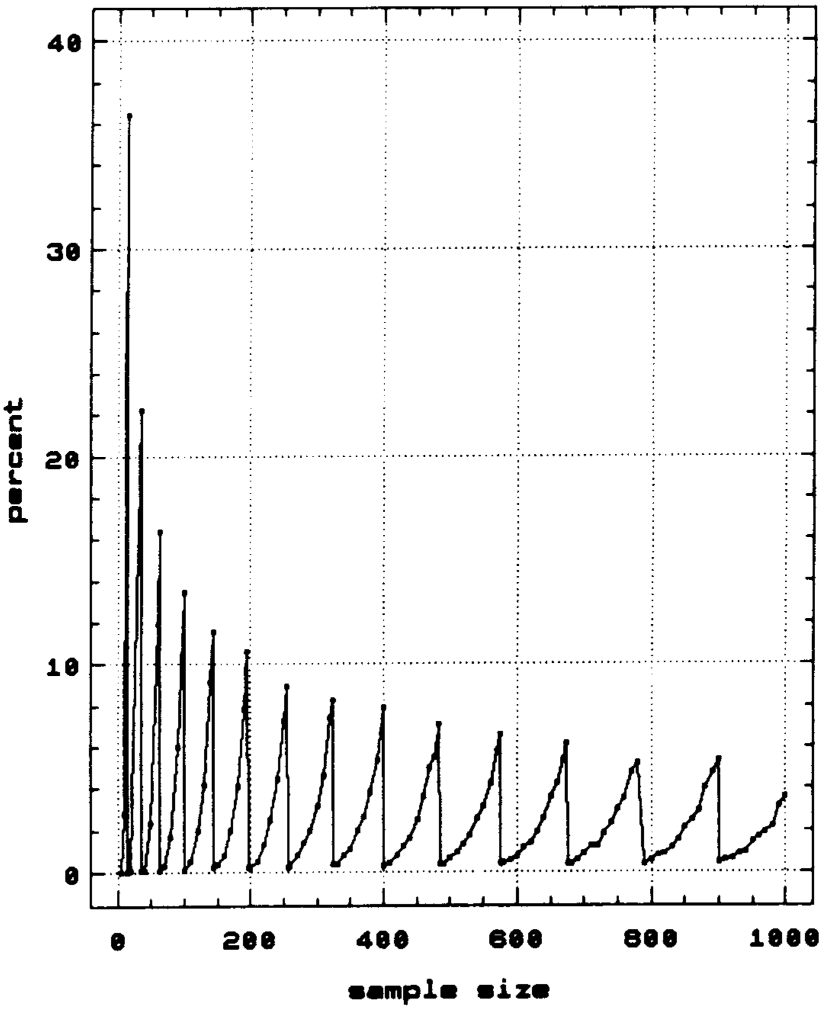


Table I. Critical values for the length tests

		5% level		1% level	
size n+1	r	smoothed length test	length test	smoothed length test	length tests
6	1	1.087	1.608	1.140	1.653
7	1	1.153	1.605	1.201	1.645
8	1	1.198	1.602	1.244	1.641
9	1	1.233	1.599	1.276	1.634
10	1	1.260	1.597	1.299	1.632
11	1	1.282	1.595	1.319	1.629
12	1	1.299	1.593	1.334	1.624
13	1	1.314	1.591	1.347	1.622
14	1	1.326	1.590	1.358	1.618
15	1	1.337	1.589	1.366	1.616
16	2	1.164	1.587	1.198	1.614
17	2	1.181	1.586	1.214	1.612
18	2	1.198	1.585	1.230	1.610
19	2	1.212	1.583	1.242	1.608
20	2	1.225	1.582	1.254	1.606

Table I. Critical values for the length tests

		5% level		1% level	
size n+1	r	smoothed length test	length test	smoothed length test	length tests
21	2	1.235	1.582	1.263	1.604
22	2	1.246	1.581	1.273	1.603
23	2	1.255	1.580	1.281	1.602
24	2	1.264	1.579	1.289	1.600
25	2	1.272	1.578	1.297	1.599
26	2	1.279	1.578	1.303	1.598
27	2	1.286	1.577	1.309	1.597
28	2	1.292	1.576	1.313	1.596
29	2	1.298	1.576	1.320	1.595
30	2	1.303	1.575	1.324	1.594
31	2	1.308	1.575	1.328	1.593
32	2	1.313	1.574	1.333	1.592
33	2	1.317	1.574	1.336	1.592
34	2	1.321	1.573	1.340	1.591
35	2	1.325	1.573	1.343	1.591
36	3	1.248	1.572	1.267	1.589
37	3	1.253	1.572	1.273	1.589
38	3	1.258	1.572	1.277	1.588
39	3	1.263	1.571	1.282	1.587
40	3	1.268	1.571	1.286	1.586
41	3	1.272	1.570	1.290	1.586
42	3	1.276	1.570	1.294	1.586
43	3	1.280	1.570	1.297	1.585
44	3	1.284	1.570	1.301	1.585
45	3	1.287	1.569	1.305	1.584
46	3	1.291	1.569	1.308	1.583
47	3	1.294	1.569	1.310	1.583
48	3	1.297	1.568	1.313	1.583
49	3	1.300	1.568	1.316	1.583
50	3	1.303	1.568	1.318	1.582
51	3	1.306	1.567	1.321	1.581
52	3	1.309	1.567	1.324	1.581
53	3	1.311	1.567	1.326	1.580
54	3	1.313	1.567	1.328	1.580
55	3	1.316	1.566	1.331	1.579
56	3	1.318	1.566	1.332	1.579
57	3	1.320	1.566	1.334	1.579
58	3	1.322	1.566	1.336	1.579
59	3	1.325	1.566	1.338	1.579
60	3	1.326	1.565	1.340	1.578
61	3	1.328	1.565	1.342	1.578
62	3	1.330	1.565	1.343	1.578
63	3	1.332	1.565	1.345	1.577
64	4	1.288	1.565	1.301	1.577
65	4	1.290	1.564	1.303	1.576
66	4	1.293	1.564	1.306	1.576
67	4	1.295	1.564	1.308	1.576
68	4	1.297	1.564	1.310	1.576
69	4	1.299	1.564	1.312	1.575
70	4	1.301	1.564	1.314	1.575
71	4	1.303	1.563	1.315	1.575
72	4	1.305	1.563	1.317	1.574
73	4	1.306	1.563	1.319	1.574

Table I. Critical values for the length tests

		5% level		1% level	
size n+1	r	smoothed length test	length test	smoothed length test	length tests
74	4	1.308	1.563	1.320	1.574
75	4	1.310	1.562	1.322	1.574
76	4	1.312	1.562	1.324	1.574
77	4	1.313	1.562	1.325	1.573
78	4	1.315	1.562	1.327	1.573
79	4	1.316	1.562	1.328	1.573
80	4	1.318	1.562	1.329	1.573
81	4	1.319	1.562	1.331	1.572
82	4	1.321	1.562	1.332	1.572
83	4	1.322	1.561	1.333	1.572
84	4	1.324	1.561	1.334	1.572
85	4	1.325	1.561	1.336	1.572
86	4	1.326	1.561	1.337	1.571
87	4	1.327	1.561	1.338	1.571
88	4	1.329	1.561	1.339	1.571
89	4	1.330	1.561	1.340	1.571
90	4	1.331	1.561	1.342	1.571
91	4	1.332	1.561	1.342	1.571
92	4	1.333	1.560	1.343	1.571
93	4	1.334	1.560	1.344	1.571
94	4	1.336	1.560	1.346	1.570
95	4	1.337	1.560	1.346	1.570
96	4	1.338	1.560	1.347	1.570
97	4	1.339	1.560	1.348	1.570
98	4	1.340	1.560	1.349	1.569
99	4	1.341	1.560	1.350	1.569
100	5	1.312	1.560	1.322	1.569
105	5	1.318	1.559	1.328	1.568
110	5	1.323	1.558	1.332	1.568
115	5	1.328	1.558	1.337	1.567
120	5	1.332	1.558	1.341	1.567
125	5	1.336	1.557	1.344	1.566
130	5	1.340	1.557	1.348	1.565
135	5	1.343	1.557	1.351	1.565
140	5	1.347	1.556	1.354	1.565
145	6	1.329	1.556	1.337	1.564
150	6	1.332	1.556	1.340	1.564
155	6	1.336	1.556	1.343	1.563
160	6	1.338	1.555	1.346	1.563
165	6	1.341	1.555	1.348	1.563
170	6	1.344	1.555	1.351	1.562
175	6	1.346	1.554	1.353	1.562
180	6	1.349	1.554	1.355	1.561
185	6	1.351	1.554	1.357	1.561
190	6	1.353	1.554	1.359	1.561
195	6	1.355	1.554	1.361	1.561
200	7	1.342	1.554	1.348	1.560
205	7	1.344	1.554	1.350	1.560
210	7	1.346	1.553	1.352	1.560
215	7	1.347	1.553	1.353	1.559
220	7	1.349	1.553	1.355	1.559
225	7	1.351	1.553	1.356	1.559
230	7	1.353	1.553	1.358	1.559

Table I. Critical values for the length tests

		5% level		1% level	
size n+1	r	smoothed length test	length test	smoothed length test	length tests
235	7	1.354	1.553	1.360	1.558
240	7	1.356	1.552	1.361	1.558
245	7	1.357	1.552	1.363	1.558
250	7	1.359	1.552	1.364	1.558
255	7	1.360	1.552	1.365	1.558
260	8	1.350	1.552	1.355	1.558
265	8	1.351	1.552	1.356	1.558
270	8	1.353	1.552	1.358	1.558
275	8	1.354	1.552	1.359	1.557
280	8	1.355	1.551	1.360	1.557
285	8	1.356	1.551	1.361	1.557
290	8	1.358	1.551	1.363	1.557
295	8	1.359	1.551	1.363	1.557
300	8	1.360	1.551	1.365	1.556
310	8	1.362	1.551	1.366	1.556
320	8	1.364	1.551	1.368	1.556
330	9	1.357	1.550	1.361	1.555
340	9	1.359	1.550	1.363	1.555
350	9	1.361	1.550	1.365	1.555
360	9	1.362	1.550	1.367	1.555
370	9	1.364	1.550	1.368	1.555
380	9	1.366	1.550	1.370	1.554
390	9	1.367	1.550	1.371	1.554
400	10	1.361	1.549	1.365	1.554
410	10	1.363	1.549	1.366	1.554
420	10	1.364	1.549	1.368	1.554
430	10	1.366	1.549	1.369	1.553
440	10	1.367	1.549	1.370	1.553
450	10	1.368	1.549	1.372	1.553
460	10	1.369	1.549	1.373	1.553
470	10	1.370	1.549	1.374	1.553
480	10	1.371	1.549	1.375	1.553
490	11	1.366	1.548	1.370	1.553
500	11	1.367	1.548	1.371	1.552

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