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Discretization of markovian queueing systems

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Abstract

Recently it turned out, that discretizing the time in a markovian queueing model makes it possible to apply powerfull combinatorial methods which often yield surprisingly simple answeres to complicated questions. In this paper we show that the continuous time solution of a markovian queueing model may be obtain from the solution of its discrete time analogue by a simple limiting procedure. Under mild regularity conditions these limiting forms can be shown to be the unique solutions of Kolmogorov's backward differential equations. Furthermore some additional methodological results concerning taboo probabilities and first passage densities are obtained. In a final section some examples are given.

1 Introduction

One of the most interesting questions arising in the study of markovian queueing systems is their transient behaviour which may be described either by the waiting time process or equivalently by the state- or queue length process $Q(t)$ ($Q(t)$ being the number of customers in the system at time t, $t \geq 0$).

Unfortunately the derivation of the distribution of $Q(t)$ is in general very complicated and may be quite intractable in many cases of partical interest, like M/M/1 queues controlled by N-policy or M/M/1 queues involving batches. But recently it has been argued by Mohanty and Panny (1990a, 1990b), Böhm and Mohanty (1990a, 1990b, 1990c, 1990d), Kanwar Sen and Jain (1990), Kanwar Sen, Jain and Gupta (1990) and Mohanty, Parthasaraty and Sharaf Ali (1990), that it may be more natural and even more useful instead of considering $Q(t)$ to consider the discrete time analogue $Q(n)$. The discrete time state process $Q(n)$ is usually constructed by splitting the interval $(0, t)$ into n time slots of equal length $h = t/n$. This discretization allows the application of combinatorial methods, like lattice path counting, and discrete time renewal theory. Using these methods it was possible to derive the distribution of $Q(n)$ in certain cases, we like to mention N-policy and queues involving batches. And it turned out that the methodology and the solutions are surprisingly simple. So these successes are a good motivation for further intensive study of the transient behaviour of discrete time markovian queueing systems.

However there remains an open problem: once we have the solution in discrete time, is it possible to find the solution in continuous time by an appropriate limiting procedure? In the cited papers various Poisson type limit theorems are given (Poisson type in the sense that while keeping the process intensities, like arrival and departure rates, fixed let $n \to \infty$). It is not immediately clear that the limiting forms obtained in this way are exactly the solutions of the corresponding continuous time queueing problem. In the simple M/M/1 case this equivalence has been shown directly by comparing the known continuous time solution with the limiting form of the discrete time result (Mohanty and Panny (1990a, 1990b), Kanwar Sen and Jain (1990), Mohanty, Parthasarty and Sharaf Ali (1990)). For M/M/1 queues under N-policy this equivalence has been proved in the special case $N = 1$ by Böhm and Mohanty (1990b).

So it is clear that the situation is quite unsatisfactory. In particular the solution of this problem requires the following:

1. a rigorous justification of the discretization procedure

2. a general proof of the equivalence of limiting forms of the discrete time distributions and their continuous time counterparts.

The answeres to these questions together with certain regularity conditions will be given in section 2. Interestingly it will turn out that our equivalence proof offers a bit more: it is constructive in the sense that it provides us with a general technique for obtaining the limiting forms. This technique, limiting forms of taboo probabilities, first passage densities and the limiting behaviour of convolution type expressions, which play an important role in applications, are discussed in section 3. Section 4 is devoted to examples, in particular we will apply the methods developed in section 3 to certain special cases: simple M/M/1 and M/M/1 under N-policy.

2 The equivalence proof

The simplest form of a markovian queueing system may be defined as an aggregate consisting of a service facility with one or more channels, the service times being independent exponentially distributed random variates and an arrival stream where the interarrival times are again independent exponentially distributed. The system opens at time $t = 0$ with $m \geq 0$ customers waiting.

Let $Q(t)$ be the state process, i.e. the number of customers in the system at time t. Then $Q(t)$ is a markov pure jump process with state space $\mathsf{N} = 0, 1, 2, \ldots$ or a finite subset thereof. This informal definition and all results below extend readily to more complicated systems. For instance there may be different types, say ν in number, of customers, differing in their arrival and/or departure rates. Then the state process is a vector valued markov pure jump process with state space N^ν, the set of all $\nu-$tuples over N. Or we may have more than one service facility arranged in a line, thus giving rise to a tandem system with finite or infinite buffer sizes. Again the state process is a vector valued markov pure jump process with state space N^ν, ν being the number of nodes. Even more complicated system may be considered, provided the topology of the system allows us to model the state process as a markov pure jump process with a denumerable state space.

However for simplicity we confine ourselves to simple systems, but we bear in mind that the crucial assumption being the denumerability of the state space.

Let $\mathcal{S}$ be the state space of the markov pure jump process $Q(t), t \geq 0$. The strong markov property implies: if $Q(\tau) = i \in \mathcal{S}$ for some time τ, then $Q(t)$ will remain constant for an interval $\tau \leq t < \tau + \mathbf{T}$, whose duration $\mathbf{T}$ has an exponential density with mean $1/c_i$. At the end of this interval $Q(t)$ jumps instananously to state $j \in \mathcal{S}$ with probability p_{ij}, where $p_{ii} = 0$ by definition.

Now define the transition probabilities $P_{mk}(t) = P(Q(t) = k|Q(0) = m)$, the vector $\mathbf{P}'_k(t) = ((P_{0k}(t), P_{1k}(t), \ldots)$, the matrix of the instananous transition probabilities $\mathbf{P} = (p_{ij})$, $\sum_j p_{ij} = 1$, and the diagonal matrix $\mathbf{C} = \text{diag}(c_0, c_1, \ldots)$. All the matrices and vectors just defined will be of infinite dimension in general. For the matrix $\mathbf{C}$ we will assume additionally that the sequence $\{c_i\}$ is bounded, say $c_i \leq M < \infty$. As a consequence of this additional assumption, it can be shown (Feller 1971, p 487, Corollary 1) that the transition probabilities are the unique solutions of Kolmogorov's forward and backward differential equations. Thus for our purpose it will be sufficient to consider the backward equations, which read as

$$\frac{d}{dt}\mathbf{P}_k(t) = \mathbf{W}\mathbf{P}_k(t) \tag{1}$$

with boundary condition $\mathbf{P}_k(0) = \mathbf{e}_k$, the $k-$th unitvector, and $\mathbf{W} = \mathbf{C}(\mathbf{P} - \mathbf{I})$, $\mathbf{I}$ being the identity matrix. To study the differential equation (1) it will be approriate to consider the linear mapping induced by the (in general) infinite dimensional matrix $\mathbf{W}$. For this purpose let $\mathcal{X}$ be the space of all bounded real valued sequences $\{u_n\}$ equipped with the norm

$$||u|| = \sup |u_i|$$

The space $\mathcal{X}$ is complete and therefore a Banach space.

Let $\mathcal{B}(\mathcal{X})$ be the set of all bounded linear operators on $\mathcal{X}$ to itself. The matrix $\mathbf{W}$ gives rise to a linear transformation $T \in \mathcal{B}(\mathcal{X})$, for let $u \in \mathcal{X}$, then the $i-$th term of Tu is given by

$$(Tu)_i = \sum_k (\mathbf{W})_{ik} u_k$$

Hence (1) may be interpreted as a special case of the differential equation

$$\frac{d}{dt}u = Tu \tag{2}$$

whose solution is given formally by

$$u(t) = e^{tT}u(0)$$

where the exponential function may be defined by its Taylor series

$$e^{tT} = \sum_{i \geq 0} \frac{(tT)^i}{i!}$$

However, the exponential function may be defined alternatively by

$$e^{tT} = \lim_{n \to \infty} (I + \frac{t}{n} T)^n \tag{3}$$

where I is the identity operator.

The interesting features of (3) are that it gives rise to a solution of (2) and is in fact *the* discretization procedure linking the processes $Q(n)$ and $Q(t)$ together in a one to one fashion.

To see this define now the linear operator

$$V_n(t) = I + \frac{t}{n} T \qquad t \geq 0,$$

which acts as follows:

$$\begin{aligned} (V_n u)_i &= ((I + \frac{t}{n} T)u)_i \\ &= \sum_k (\mathbf{I} + \frac{t}{n} \mathbf{C}(\mathbf{P} - \mathbf{I}))_{ik} u_k \end{aligned} \tag{4}$$

Then the following lemma holds:

Lemma 1 *If $c_i \leq M < \infty$, then for all $n \geq tM$ and $0 \leq t < \infty$ $V_n(t)$ is a transition operator.*

Proof. First we have to show, that $V_n(t)$ is positive for $n \geq tM$. From the matrix representation (4) it follows that for $i \neq k$ nonnegativity is automatically satisfied and for $i = k$, the inequality $1 - \frac{t}{n} c_i \geq 0$ requires for all i that $n \geq tM$. Next we have to show that $V_n(t)\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the sequence, all whose terms are equal to 1. Clearly $\mathbf{1} \in \mathcal{X}$. This is equivalent to show that the row sums of $\mathbf{I} + \frac{t}{n}\mathbf{C}(\mathbf{P} - \mathbf{I})$ are equal to one. But

$$\begin{aligned} \sum_j (\delta_{ij} + \frac{t}{n} c_i (p_{ij} - \delta_{ij})) &= 1 + \frac{t}{n} c_i \sum_j (p_{ij} - \delta_{ij}) \\ &= 1 \end{aligned}$$

since $\sum_j p_{ij} = 1$. Furthermore $\|V_n(t)u\| \leq \|u\|$ for $n \geq tM$ and thus $\|V_n(t)\| \leq 1$.

Consider now the operator $A_n(t) = V_n^n(t)$. Since $\|V_n(t)\| \leq 1$ it follows that $A_n(t)$ is bounded. Moreover $A_n(t)$ is strongly continuous,

$$A_n(t) \rightarrow A_n(0) = I \qquad \text{as} \qquad t \rightarrow 0$$

Let us now show that $A_n(t)u, u \in \mathcal{X}$ forms a Cauchy sequence. For this purpose we note that

$$\frac{d}{dt}A_n(t) = T(I + \frac{t}{n}T)^{n-1}$$

and

$$\begin{aligned} [A_n(t) - A_m(t)]u &= \lim_{\epsilon \rightarrow 0+} \int_\epsilon^{t-\epsilon} \frac{d}{ds}[A_m(t-s)A_n(s)u]\, ds \\ &= \int_0^t T^2 \left(\frac{t-s}{m} - \frac{s}{n}\right)\left(I + \frac{s}{n}T\right)^{n-1}\left(I + \frac{t-s}{m}T\right)^{m-1} u\, ds \end{aligned}$$

Since $\|A_n(t)\| \leq 1$ for all $m, n \geq tM$ and T is bounded, we obtain the estimate:

$$\begin{aligned} \|A_n(t)u - A_m(t)u\| &\leq \|T^2 u\| \int_0^t \left(\frac{t-s}{m} + \frac{s}{n}\right) ds \\ &= \frac{t^2}{2}\left(\frac{1}{m} + \frac{1}{n}\right)\|T\|^2\|u\| \end{aligned}$$

Now since $A_n(t)$ is uniformly bounded we conclude that the strong limit of $A_n(t)$ exists and define the function

$$\begin{aligned} U(t) &= e^{tT} \\ &= \text{s-}\lim_{n \rightarrow \infty} A_n(t) \end{aligned} \tag{5}$$

As a side result we obtain the following error estimate:

$$\|e^{tT}u - A_n(t)u\| \leq \frac{t^2}{n}M\|u\|$$

Let us now show that $U(t)$ is in fact a solution of (2). We already know that

$$\begin{aligned} \frac{d}{dt}A_n(t) &= T(I + \frac{t}{n})^{n-1} \\ &= T(I + \frac{t}{n}T)^{-1}A_n(t) \end{aligned}$$

Now $T(I + \frac{t}{n}T)^{-1}u \to Tu$, since $A_n(t)$ is strongly continuous. Furthermore

$$A_n(t)u - A_n(0)u = \int_0^t \frac{d}{ds}A_n(s)u\,ds$$

and therefore

$$A_n(t)u - u = \int_0^t (I + \frac{s}{n}T)^{n-1}Tu\,ds$$

But $(I + \frac{s}{n}T)^{n-1} = (I + \frac{s}{n}T)^{-1}A_n(s) \to U(s)$ uniformly in s, thus passing to the limit we find:

$$U(t)u - u = \int_0^t U(s)Tu\,ds$$

and after differentiating

$$\frac{d}{dt}U(t)u = U(t)Tu$$

Hence $u(t) = U(t)u(0)$ is in fact a solution of (2). It can be shown that this solution is also unique (Kato (1966), p. 481).

Now the discretization procedure may be described as follows: split the interval $(0,t)$ into $n, n \geq tM$ slots of equal length $h = t/n$ and define the discrete time analogue of $Q(t)$ by $Q(n)$, where for every m, $0 \leq m \leq n$

$$Q(m) = Q(\tau) \qquad \text{for} \quad \frac{(m-1)t}{n} < \tau \leq \frac{mt}{n}$$

The resulting markov chain is governed by the one-step transition operator $V_n(t)$ with matrix representation $\mathbf{I} + \frac{t}{n}\mathbf{C}(\mathbf{P} - \mathbf{I})$. Hence the $n-$step transition probabilities or equivalently the distribution of $Q(n)$ are determined by $A_n(t)$ with matrix representation $(\mathbf{I} + \frac{t}{n}\mathbf{C}(\mathbf{P} - \mathbf{I}))^n$. Thus once we know the $n-$step transition probabilities we may pass to the limit $n \to \infty$, this limit is well defined and is the unique solution of the original continuous time problem.

3 Technical implications

3.1 A useful theorem

One might have got the impression that even if we know that

$$\lim_{n\to\infty} P(Q(n) = k|Q(0) = m) = P(Q(t) = k|Q(0) = m)$$

the derivation of this limit may be difficult in practice. However, this is not the case.

Let $n \geq tM$, then $\mathbf{Q} = \mathbf{I} + \frac{t}{n}\mathbf{C}(\mathbf{P} - \mathbf{I})$ is the transition matrix of the associated markov chain. If we can determine the $n-$step transition probabilities, or what amounts to the same $\mathbf{Q}^n$, then this information will be sufficient for deriving the solutions of equation (1). These solutions are given by

$$\mathbf{P}_k(t) = e^{\mathbf{W}t}\mathbf{P}_k(0)$$

and since the linear transformation induced by $\mathbf{W}$ is bounded, the Taylor series expansion of $e^{\mathbf{W}t}$ is well defined. Now

$$\begin{aligned} \mathbf{W}t &= t\mathbf{C}(\mathbf{P} - \mathbf{I}) \\ &= n(\mathbf{Q} - \mathbf{I}) \end{aligned}$$

Hence

$$e^{\mathbf{W}t} = e^{-n}e^{n\mathbf{Q}} \tag{6}$$

Note that this equation is an identity. It holds for all n and $t \in [0, \infty)$ and actually is independent of n by construction of $\mathbf{Q}$. Using this relation we find

$$e^{\mathbf{W}t} = e^{-n}\sum_{k\geq 0}\frac{n^k}{k!}\mathbf{Q}^k$$

Now let $q_{ij}^{(k)}$ be the $ij-$th entry of $\mathbf{Q}^k$ and define the pgf.

$$\phi_{ij}(s) = \sum_{k\geq 0} s^k q_{ij}^{(k)}$$

Then the following theorem establishes the relation between $\phi_{ij}(s)$ and the corresponding continuous time solution:

Theorem 1

$$(e^{\mathbf{W}t})_{ij} = [s^0]\phi_{ij}(s)e^{-n+n/s} \tag{7}$$

where $[s^k]$ is the coefficient operator.

Proof:

$$\begin{aligned}
(e^{\mathbf{W}t})_{ij} &= e^{-n}\sum_{k\geq 0}\frac{n^k}{k!}q_{ij}^{(k)} \\
&= e^{-n}\sum_{k\geq 0}\frac{n^k}{k!}[s^k]\phi_{ij}(s) \\
&= [s^0]\phi_{ij}(s)e^{-n}\sum_{k\geq 0}\frac{n^k s^{-k}}{k!} \\
&= [s^0]\phi_{ij}(s)e^{-n+n/s}
\end{aligned}$$

3.2 Taboo probabilitites, first passages and convolutions

Let H be a finite subset of S and define the taboo probabilitites

$${}_HP(Q(t)=k|Q(0)=m) = P(Q(t)=k, Q(s)\notin H, 0<s<t|Q(0)=m)$$

where $m,k\notin H$. It can be shown that these taboo probabilitites satisfy the same forward and backward equations as $P(Q(t)=k|Q(0)=m)$ with the modifications that $c_k=0$ and $p_{kj}=0$ for $k\in H$ (Feller 1971, p. 494). Thus our results from section 2 apply without change and we conclude that

$${}_HP(Q(n)=k|Q(0)=m) \to {}_HP(Q(t)=k|Q(0)=m) \quad \text{as} \quad n\to\infty \tag{8}$$

Now let H consist of a single state, say $H=\{j\}$, and define the stopping time

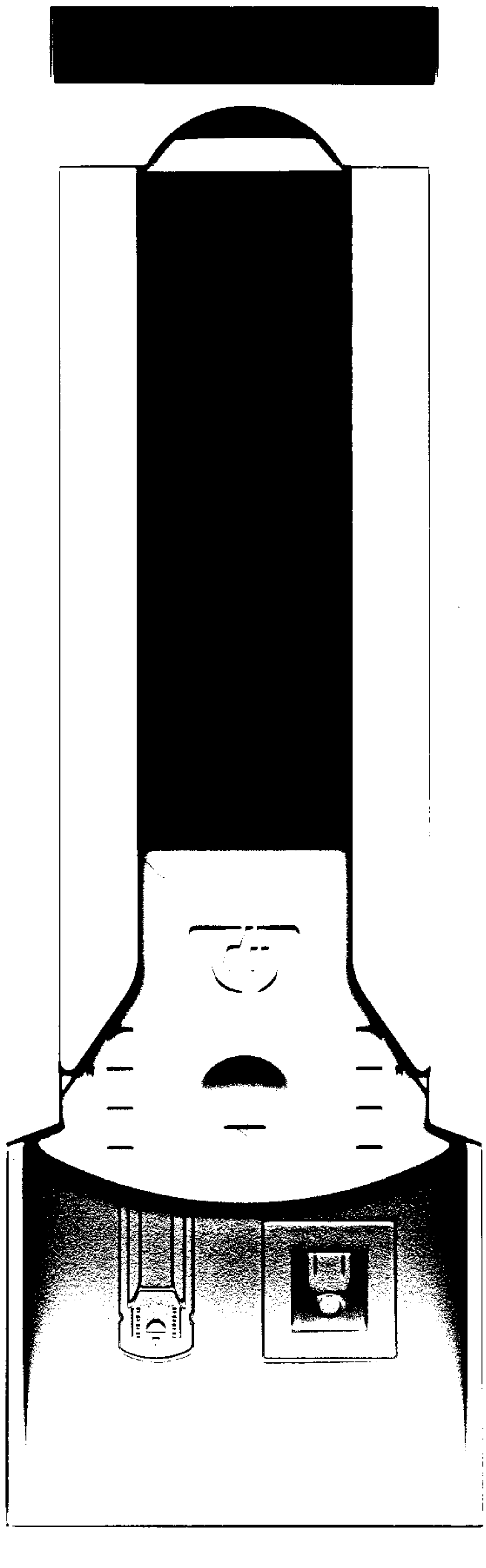
$$T_{ij} = \inf\{t : Q(t)=j|Q(0)=i\}$$

with density $f_{ij}(t)$. Then

$$\begin{aligned}
f_{ij}(t)dt &= P(t\leq T_{ij}<t+dt) \\
&= \sum_{k\neq j} {}_jP(Q(t)=k|Q(0)=m)w_{kj}\,dt
\end{aligned}$$

where $w_{kj}=(\mathbf{W})_{kj}$. Analogously in discrete time we define

$$N_{ij} = \inf\{n : Q(n)=j|Q(0)=i\}$$



and let $g_{ij}(n)$ be its probability function. Then

$$\begin{aligned} g_{ij}(n) &= P(n-1 < N_{ij} \le n) \\ &= P(N_{ij} = n) \\ &= \sum_{k \neq j} {}_jP(Q(n-1) = k | Q(0) = m) q_{kj} \\ &= \sum_{k \neq j} {}_jP(Q(n-1) = k | Q(0) = m) w_{kj} \frac{t}{n} \end{aligned}$$

where $q_{kj} = (\mathbf{Q})_{kj}$. And therefore

$$g_{ij}(n)\frac{n}{t} = \sum_{k \neq j} {}_jP(Q(n-1) = k | Q(0) = m) w_{kj} \tag{9}$$

Hence in the limit, using (8) and uniform convergence of the series an the right hand side of (9):

$$\begin{aligned} \lim_{n\to\infty} g_{ij}(n)\frac{n}{t} &= \lim_{n\to\infty} \sum_{k \neq j} {}_jP(Q(n-1) = k | Q(0) = m) w_{kj} \\ &= \sum_{k \neq j} {}_jP(Q(t) = k | Q(0) = m) w_{kj} \\ &= f_{ij}(t) \end{aligned}$$

Quite often one encounters the problem of determining the probability that the process $Q(t)$, starting in m, reaches at time t state k and has a first passage through state j at some time $0 < s < t$. In particular we define

$${}^jP(Q(t) = k | Q(0) = m) = \int_0^t f_{mj}(s) P(Q(t-s) = k | Q(0) = j)\, ds \tag{10}$$

The discrete time analogue is given by

$${}^jP(Q(n) = k | Q(0) = m) = \sum_{\nu} g_{mj}(\nu) P(Q(n-\nu) = k | Q(0) = j) \tag{11}$$

We will now show that

$$\begin{aligned} &\lim_{n\to\infty} \sum_{\nu} g_{mj}(\nu) P(Q(n-\nu) = k | Q(0) = j) = \\ &= \int_0^t f_{mj}(s) P(Q(t-s) = k | Q(0) = j)\, ds \end{aligned} \tag{12}$$

First observe that (10) implies that the state j is visited at least once. Hence we immediately have the decompositions

$$\begin{aligned} & P(Q(t)=k|Q(0)=m) = \\ & = \; {}_jP(Q(t)=k|Q(0)=m) + {}^jP(Q(t)=k|Q(0)=m) \end{aligned}$$

and in discrete time

$$\begin{aligned} & P(Q(n)=k|Q(0)=m) = \\ & = \; {}_jP(Q(n)=k|Q(0)=m) + {}^jP(Q(n)=k|Q(0)=m) \end{aligned}$$

But since

$$\begin{aligned} P(Q(n)=k|Q(0)=m) & \rightarrow & P(Q(t)=k|Q(0)=m) \\ {}_jP(Q(n)=k|Q(0)=m) & \rightarrow & {}_jP(Q(t)=k|Q(0)=m) \end{aligned}$$

as $n \to \infty$, (12) follows.

4 Examples

4.1 Simple M/M/1

In this example the matrix $\mathbf{Q}$ has the following structure

$$\begin{aligned} q_{ii} &= 1-\tfrac{\lambda+\mu}{n}t && i>0 \\ q_{00} &= 1-\tfrac{\lambda}{n}t && i=0 \\ q_{ii+1} &= \tfrac{\lambda}{n}t && i\geq 0 \\ q_{ii-1} &= \tfrac{\mu}{n}t && i\geq 1 \end{aligned}$$

which follows from the well know definition of the infinitesimal generator of the system (1). For convenience we set $\alpha = \frac{\lambda t}{n}, \gamma = \frac{\mu t}{n}$ and $\beta = 1-\alpha-\gamma$. Now define $P_{mk}(t) = P(Q(t) = k|Q(0) = m)$ and its discrete time analogue $P_{mk}(n)$. Let $\phi_{mk}(s)$ be the pgf. of $P_{mk}(n)$. It has been shown be Mohanty and Panny (1990b) that

$$\phi_{mk}(s) = \frac{v^{m-k}}{\gamma}\frac{\alpha v^2+\beta v+\gamma}{1-\rho v^2}\frac{1-v+v(1-\rho v)(\rho v^2)^k}{1-v} \tag{13}$$

where the substitution $s = (\alpha v + \beta + \gamma/v)^{-1}$ has been used.

By theorem 1 we find

$$P_{mk}(t) = [s^0]\phi_{mk}(s)e^{-n+n/s}$$

Hence using Cauchy's integral theorem:

$$\begin{aligned}
P_{mk}(t) &= \frac{1}{2\pi i}\oint \frac{e^{-n}e^{(\lambda tv+n-(\lambda+\mu)t+\mu t/v)}v^{m-k}(1-v+v(1-\rho v)(\rho v^2)^k)}{v(1-v)} \\
&= [v^0]e^{-(\lambda+\mu)t}e^{\lambda tv+\mu t/v}\times \\
&\quad \times\sum_{i\geq 0}(v^{i+m+k}-v^{i+m-k+1}+\rho^k v^{m+k+i+1}-\rho^{k+1}v^{m+k+i+2})
\end{aligned}$$

Observing that

$$[v^k]e^{\lambda tv+\mu t/v} = \rho^{k/2}I_k(2t\sqrt{\lambda\mu}) \tag{14}$$

we find after some simplifications

$$P_{mk}(t) = e^{-(\lambda+\mu)t}\left[\rho^{\frac{k-m}{2}}I_{k-m}+\rho^{\frac{k-m-1}{2}}I_{k+m+1}+(1-\rho)\sum_{j\geq k+m+2}\rho^{\frac{2k-j}{2}}I_j\right]$$

where the abbreviation $I_k = I_k(2t\sqrt{\lambda\mu})$ has been used.

4.2 M/M/1 under N-policy

The transient solution of M/M/1 queues controlled by N-policy has been derived by purely combinatorial arguments by Böhm and Mohanty (1990a, 1990b) and Kanwar Sen, Jain and Gupta (1990). Here we will give a short on generating functions based derivation of the transient solution.

N-policy means, that whenever the queue becomes empty, the server remains closed until a new queue of prescribed length $N \geq 1$ has built up. Thus the state space may be decomposed into two subclasses, a class I corresponding to the idle state and a class B corresponding to the busy state, respectively. Clearly $I = 0, 1, \ldots, N-1$ and $B = 1, 2, 3, \ldots$. From the above definition it should be clear that the process $Q(t)$ behaves like a simple birth process with rate λ when it is in class I and it behaves like a simple birth - death process with rate λ and

μ when it is in class B. Let ${}_{BB}^{\,i}P_{mk}(t)$ denote the probability, that the server is initially busy with m customers waiting and is busy at time t for the i–the time (not counting the initial left censored busy period) with k customers in the system. A typical sample path looks like Figure 1.

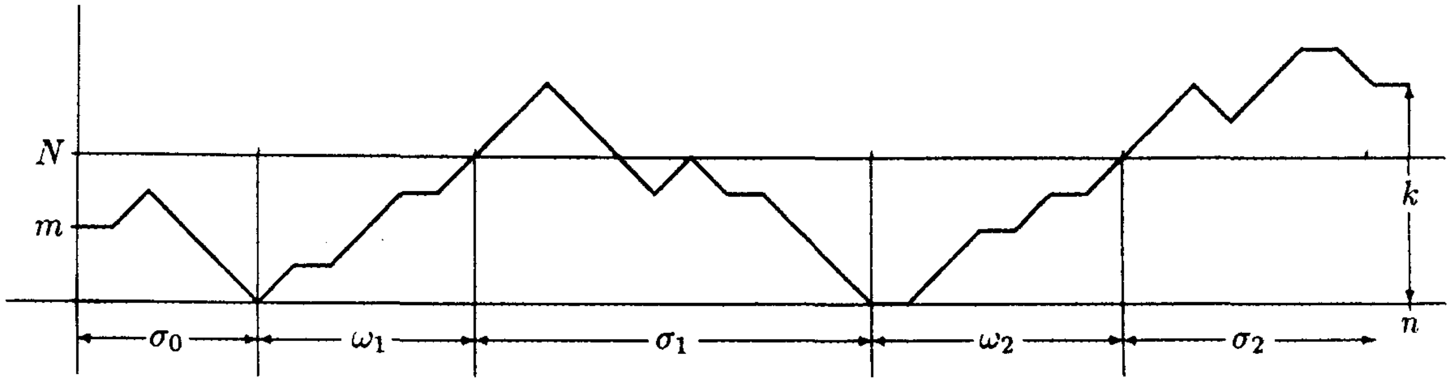


Fig. 1

Figure 1, where σ_i (ω_i) denotes the i–th busy (idle) period, reveals the renewal properties of the process. Renewal epochs are those time points, where the server becomes busy (idle) for the first time (first passages through 0 and N).

Let us discretize the model and use the same abbreviations (α, β, γ) as in the previous example. Furthermore let ${}_{BB}^{\,i}\mathcal{P}_{mk}(s)$ denote the pgf. of ${}_{BB}^{\,i}P_{mk}(n)$, the discrete time analogue of ${}_{BB}^{\,i}P_{mk}(t)$.

Now define the following probabilities and their pgfs:

$A_n(m) = P($a discrete time birth-death process has a first passage through 0, provided it started in $m)$, with pgf. $\mathcal{A}_m(s)$.

$B_n(m,k) = P($a discrete time birth-death process terminates in k at time n without visiting state 0, provided it started in $m)$, with pgf. $\mathcal{B}_{m,k}(s)$.

$C_n(N) = P($a discrete time birth process has a first passage through N, provided it started in 0$)$, with pgf. $\mathcal{C}_N(s)$.

Then

$$ {}_{BB}^{\,i}\mathcal{P}_{mk}(s) = \mathcal{A}_m(s)\mathcal{A}_N^{i-1}(s)\mathcal{B}_{Nk}(s)\mathcal{C}_N(s) \tag{15} $$

Now observe that

$$ C_n(N) = \binom{n-1}{N-1}\alpha^N(1-\alpha)^{n-N}, $$

and the i–fold convolution of $\mathcal{C}_N(s)$ is the pgf. of $C_n(Ni)$.
Let $f_{0N}(t) = \lim_{n\to\infty} C_n(Ni)n/t$, then we find immediately

$$f_{0N}(t) = \frac{\lambda^{Ni}}{(Ni-1)!} t^{Ni-1} e^{-\lambda t}$$

The pgfs $\mathcal{A}$ and $\mathcal{B}$ are well known and may be found for instance in Panny (1984):

$$\mathcal{A}_m(s) = v^m \qquad \text{where} \qquad s = v/(\alpha v^2 + \beta v + \gamma)$$

and

$$\mathcal{B}_{Nk}(s) = \frac{v^{|k-N|}}{\gamma - \alpha v^2} \rho^{\frac{|k-N|+k-N}{2}} (\alpha v^2 + \beta v + \gamma)(1 - (\rho v^2)^{\frac{N+k-|k-N|}{2}})$$

Thus (15) simplifies to

$$\begin{aligned}
{}_{BB}^{\,i}\mathcal{P}_{mk}(s) &= \frac{v^{m+N(i-1)+|k-N|}}{\gamma - \alpha v^2} \rho^{\frac{|k-N|+k-N}{2}} \times \\
&\quad \times (\alpha v^2 + \beta v + \gamma)(1 - (\rho v^2)^{\frac{N+k-|k-N|}{2}}) \mathcal{C}_{Ni}(s) \\
&= \Phi_{mk}(s)\mathcal{C}_{Ni}(s)
\end{aligned}$$

where $\Phi_{mk}(s) = \sum s^n \phi_{mk}(n)$. Let $\phi_{mk}(t)$ be the continuous time counterpart of $\phi_{mk}(n)$. Then by theorem 1:

$$\begin{aligned}
\phi_{mk}(t) &= [s^0]\Phi_{mk}(s)e^{-n+n/s} \\
&= \frac{1}{2\pi i} e^{-(\lambda+\mu)t} \rho^{\frac{|k-N|+k-N}{2}} \times \\
&\quad \times \oint v^{m+N(i-1)+|k-N|-1}(1 - (\rho v^2)^{\frac{N+k-|k-N|}{2}}) e^{\lambda v t + \mu t/v}\, dv
\end{aligned}$$

Using (14) we get

$$\begin{aligned}
{}_{BB}^{\,i}P_{mk}(t) &= \qquad\qquad (16) \\
&= \int_0^t \phi_{mk}(s) f_{0N}(t-s)\, ds \\
&= \rho^{\frac{k-m-Ni}{2}} e^{-(\lambda+\mu)t} \int_0^t \frac{e^{-\lambda(t-s)}(t-s)^{Ni-1}}{(Ni-1)!} [I_{m+N(i-1)+|k-N|} - I_{m+Ni+k}]\, ds
\end{aligned}$$

where I_k is an abbreviation of $I_k(2s\sqrt{\lambda\mu})$.

5 References

[1] Böhm W., Mohanty S.G.(1990a) On the transient solution of N-policy queues. Technical report Mathematics and Statistics Department, McMaster University, Hamilton.

[2] Böhm W., Mohanty S.G.(1990b) The transient solution of M/M/1 queues under (M,N)-policy. A combinatorial approach. To appear in the Journal of Statistical Planning and Inference.

[3] Böhm W., Mohanty S.G.(1990c) On random walks with barriers and their application to queues. Technical report Mathematics and Statistics Department, McMaster University, Hamilton.

[4] Böhm W., Mohanty S.G.(1990d) On M/M/1 queues involving batches. Technical Report, Department of Mathematical Statistics, University of Economics, Vienna.

[5] Feller W. (1971) *An Introduction to Probability Theory and Its Applications*, Vol. 2, 2nd Ed., New York, John Wiley and Sons.

[6] Kanwar Sen, Jain J.L., Gupta J.M. (1990) Lattice path approach to transient solution of M/M/1 with (0,K) control policy. Paper presented at the 2nd. Conference on Lattice Path Combinatorics and Applications, July 5-7 1990, Hamilton, Ontario.

[7] Kanwar Sen, Jain J.L. (1990) Combinatorial approach to markovian queues. Paper presented at the 2nd. Conference on Lattice Path Combinatorics and Applications, July 5-7 1990, Hamilton, Ontario.

[8] Kato T, (1966) *Perturbation Theory for Linear Operators.* New York, Springer.

[9] Mohanty S.G., Panny W. (1990a) A discrete time analogue of the M/M/1 queue and the transient solution: a geometric approach. Sankyā, Ser. A, to appear.

[10] Mohanty S.G., Panny W. (1990b) A discrete time analogue of the M/M/1 queue and the transient solution. Proceedings of the 3rd. Hungarian Colloquium on Limit Theorems in Probability and Statistics, P. Révész ed., Amsterdam: North Holland (1990).

[11] Mohanty S.G., Parthasarathy P.R., Sharaf Ali M. (1990) On the transient solution of a discrete queue. Submitted for publication.