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On Two-Periodic Random Walks with Boundaries

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Abstract. Two-periodic random walks are models for the one-dimensional motion of particles in which the jump probabilities depend on the parity of the currently occupied state. Such processes have interesting applications, for instance in chemical physics where they arise as embedded random walk of a special queueing problem. In this paper we discuss in some detail first passage time problems of two-periodic walks, the distribution of their maximum and the transition functions when the motion of the particle is restricted by one or two absorbing boundaries. As particular applications we show how our results can be used to derive the distribution of the busy period of a chemical queue and give an analysis of a somewhat weird coin tossing game.

Keywords: random walk, chemical queue, absorbing boundaries

AMS classification: 60G50, 60K25

1. Introduction

This paper is devoted to the study of two-periodic random walks. Such processes arise when we consider the following type of random motion of particles: particles move on the set of integers, at each epoch a particle may jump one unit to the left or to the right. The probability of a jump to the left or to the right depends on whether the position the particle currently occupies is even or odd. More formally, we consider a process \( S_n = \sum_{i=1}^{n} X_i \) with \( S_0 = 0 \) and increments \( X_i \in \{-1, 1\} \) having probabilities

\[
\begin{align*}
P(X_i = 1|S_{i-1} = 2m) &= \alpha, \\
P(X_i = -1|S_{i-1} = 2m) &= 1 - \alpha, \\
P(X_i = 1|S_{i-1} = 2m + 1) &= 1 - \alpha, \\
P(X_i = -1|S_{i-1} = 2m + 1) &= \alpha.
\end{align*}
\]

Stockmayer et al. (1977) seem to be the first to study this type of process in a chemical physics context. They considered heterogeneous chains of two regularly alternating kinds of atoms, so that the chain has the structure \( \ldots ABABA \ldots \). The atoms are joined by links which are exposed to random shocks. The latter cause atoms to move in the chain and the molecule to diffuse. It is assumed that the atoms have different jump rates, say \( \lambda \) and \( \mu \), respectively. Let \( Q(t) \) denote the
position an atom occupies within the chain at time \( t \) and let \( p_j(t) = P(Q(t) = j) \). Assuming exponentially distributed sojourn times of the atoms the \( p_j(t) \) can be found as solutions of the system of equations

\[
\frac{dp_{2j}(t)}{dt} = -(\lambda + \mu)p_{2j}(t) + \mu p_{2j-1}(t) + \lambda p_{2j+1}(t)
\]

\[
\frac{dp_{2j+1}(t)}{dt} = -(\lambda + \mu)p_{2j+1}(t) + \lambda p_{2j}(t) + \mu p_{2j+2}(t),
\]

which is a particular case of coupled Chapman-Kolmogorov equations. If no further conditions are imposed on (2) except for an initial condition like \( p_0(0) = 1 \), then the \( p_j(t) \) are transition functions of a two-periodic randomized random walk whose state space is the set of all integers. But negative positions do not make sense in a chemical context, so it is quite natural to put an impenetrable barrier at zero, which is formalized by the boundary condition

\[
\frac{dp_0(t)}{dt} = -\lambda p_0(t) + \lambda p_1(t).
\]

The latter condition makes \( Q(t) \) to a queueing process of a very special type, a chemical queue.

Chemical queueing processes have been discussed by Conolly et al. (1997) who based their analysis on the system (2) and (3). Recently Tarabia et al. (2007, 2008) gave a thorough analysis by means of a uniformization procedure. They formulated the system of partial difference equations satisfied by the transitions functions of the embedded discrete time Markov chain \( Q_n \), cleverly guessed its solutions and then provided a proof by induction.

A careful study of the papers cited above suggests that a closer examination of this type of process might be a interesting task. We think that this expectation is indeed justified as is demonstrated by the results we are going to present below.

In our work we will concentrate on the discrete time random walk which is obtained by observing \( Q(t) \) at its jump times and by neglecting the reflecting barrier at zero. This embedded random walk is then two-periodic as defined in (1) with jump probability \( \alpha = \lambda/(\lambda + \mu) \).

In section 2 we analyze first passage times of \( S_n \) through upper and lower boundaries. We find that the corresponding probability generating functions (pgf.) satisfy a system of functional equations. Explicit expressions for the first passage probabilities are obtained by multivariate Lagrange inversion and we also provide asymptotic formulas. As is well known, the distribution of first passage times is closely related to the distribution of the maximum \( M_n = \max_{0 \leq i \leq n} S_i \). With regard to \( M_n \) two-periodic walks exhibit a somewhat strange behaviour. This is easily seen when we consider \( S_n \) in cases where the jump probabilities take their extremal values \( \alpha = 0 \) and \( \alpha = 1 \), see Figure 1. In both cases the lattice paths
representing the sample paths of $S_n$ collapse to simple zig-zag lines with maxima being zero and one. Intuitively we expect that the expectation of the maximum should assume its largest value for $\alpha = 1/2$, since in this case the variance of $S_n$ is maximal. Surprisingly this is not true for finite $n$. We will find that for large values of $n$ the expectation $E(M_n)$ is maximal for

$$\alpha \sim \frac{1}{2} + \frac{1}{2} \sqrt{\frac{\pi}{\pi + 8n}}.$$

(4)

Less surprising is the fact that the limiting distribution of $M_n$ is stable of index 1/2 as in the case of simple random walks.

In section 3 we consider in more detail the transition functions of $S_n$. It is not difficult to find them by a rather straightforward combinatorial argument, but our main interest is in the corresponding generating functions which will play an essential role we when discuss $S_n$ in presence of absorbing boundaries. This is the major topic of section 4, where we derive the distribution of $S_n$ in presence of one and two absorbing boundaries. Our approach is based on combinatorial decompositions of the sample paths which map to simple algebraic manipulations of basic path generating functions. Our results show some structural similarity to the well known formulas for the simple random walk in case of one boundary. In case of two boundaries this similarity is still present but the formulas are more complicated. In section 5 we present an almost fair coin tossing game which is a paraphrase of two-periodic random walks with two absorbing boundaries. The special feature of this game is that players are tossing a biased coin. We prove that even in this setting a fair game is possible under certain conditions and derive ruin probabilities and the expected duration of the game. Finally in the last section we show how our results may be used in the discussion of the chemical queue. As a particular example we give an explicit formula for the distribution of the busy period with an arbitrary number of customers initially waiting, thereby extending an earlier result of Conolly et al. (1997).
2. First passage times and the maximum

Let \( a \in \mathbb{Z} - \{0\} \) and define random stopping times \( T_a = \inf\{n : S_n = a\} \) (recall that we assume \( S_0 = 0 \) throughout this paper). In what follows it will be convenient to define a random walk \( S_n^* \) adjoint to \( S_n \) which is the same as \( S_n \) except that the jump probabilities \( \alpha \) and \( 1 - \alpha \) are interchanged. Therefore we define also stopping times \( T_a^* = \inf\{n : S_n^* = a\} \). With these random variables we associate pgf.s

\[
\begin{align*}
F_a(s) &= \sum_{n \geq 0} s^n P(T_a = n) \quad \text{and} \\
G_a(s) &= \sum_{n \geq 0} s^n P(T_a^* = n).
\end{align*}
\]

It will simplify notation considerably, if we use the abbreviations \( F := F_1(s) \) and \( G := G_1(s) \).

Our first observation is that

\[
G_2(s) = F_2(s) = FG,
\]

and upon conditioning on the first steps of \( S_n \) and \( S_n^* \), we find that the following equations hold:

\[
\begin{align*}
F &= \alpha s + (1 - \alpha)sG_2(s) \\
G &= (1 - \alpha)s + \alpha sF_2(s),
\end{align*}
\]

which in light of (5) simplify to the system of functional equations

\[
\begin{align*}
F &= \alpha s + (1 - \alpha)sFG \\
G &= (1 - \alpha)s + \alpha sFG
\end{align*}
\]

Thus \( F \) and \( G \) are solutions of the quadratic equations

\[
\begin{align*}
\alpha s F^2 - F(1 - s^2(1 - 2\alpha)) + \alpha s &= 0 \\
(1 - \alpha)sG^2 - G(1 + s^2(1 - 2\alpha)) + (1 - \alpha)s &= 0
\end{align*}
\]

Solving (8) we obtain

\[
F = \frac{1 - s^2(1 - 2\alpha) - \sqrt{(1 - s^2(1 - 2\alpha))^2 - 4\alpha^2s^2}}{2\alpha s}.
\]

Note that we have to take here the square root with the negative sign, since the other solution does not yield a power series in \( s \). In fact, the second solution is a Laurent series with principal part equal to \( 1/(\alpha s) \).
Similarly, we obtain $G$ either by solving the second equation in (8) or by simply replacing $\alpha \to 1 - \alpha$ in $F$:

\begin{equation}
G = \frac{1 + s^2(1 - 2\alpha) - \sqrt{(1 + s^2(1 - 2\alpha))^2 - 4(1 - \alpha)^2s^2}}{2(1 - \alpha)s}.
\end{equation}

Once $F$ and $G$ are known we easily get the pgf.s $F_\alpha(s)$ and $G_\alpha(s)$. Indeed, by quasi symmetry of $S_n$ and $S^*_n$ there holds

\begin{equation}
F_\alpha(s) = G_{-\alpha}(s) \quad \text{for all } a \neq 0,
\end{equation}

and by a simple renewal argument we obtain the important functional relations

\begin{equation}
F_{2a}(s) = (FG)^a \quad \text{for all } a \neq 0,
\end{equation}

and for $a > 0$:

\begin{align}
F_{2a+1}(s) &= (FG)^a F \\
F_{-2a-1}(s) &= (FG)^a G.
\end{align}

These relations may now be used to obtain series expansions of $F_\alpha(s)$. However, these expansions are rather difficult to come by when we use directly (9) and (10). Therefore we shall pursue another route, but it should be noted that (9) and (10) still have their merits, since they give us information about the singularities of $F$ and $G$ and this information is indispensible for obtaining asymptotic approximations. We shall return to that point later.

To most efficient way to get explicit formulae for $P(T_a = n)$ is by multivariate Lagrange inversion, which is a generalization of the classical Lagrange inversion formula by Jacobi (1830) and Good (1960). For more information on this powerful technical device see also Goulden and Jackson (1983), pp. 21, Bender and Richmond (1998); an excellent treatise on that subject is Hofbauer (1982).

Let us put $\lambda_1 = F$ and $\lambda_2 = G$. Futhermore in accordance with (6) define

\begin{align}
\Phi_1 &= \Phi_1(\lambda_1, \lambda_2) = \alpha + (1 - \alpha)\lambda_1\lambda_2 \\
\Phi_2 &= \Phi_2(\lambda_1, \lambda_2) = 1 - \alpha + \alpha\lambda_1\lambda_2
\end{align}

and suppose that

\begin{align}
w_1 &= s\Phi_1(w_1, w_2) \\
w_2 &= s\Phi_2(w_1, w_2)
\end{align}

for arbitrary indeterminates $w_1$ and $w_2$. 
Then for any formal Laurent series \( f(w_1, w_2) \) the multivariate Lagrange inversion formula yields the expansion:

\[
(17) \quad f(w_1(s), w_2(s)) = \sum_{i,j} [\lambda_1^i \lambda_2^j] f(\lambda_1, \lambda_2) \Phi_1^i \Phi_2^j \det \left[ \delta_{xy} - \frac{\lambda_y}{\Phi_x} \frac{\partial \Phi_y}{\partial \lambda_y} \right], \quad x, y = 1, 2,
\]

where \( \delta_{xy} \) denotes the Kronecker delta and \([\lambda_1^i \lambda_2^j]\) is the bivariate coefficient operator. The functional determinant in (17) is easily evaluated to

\[
1 - \frac{(1 - \alpha) \lambda_1 \lambda_2}{\Phi_1} - \frac{\alpha \lambda_1 \lambda_2}{\Phi_2} = \alpha(1 - \alpha) \frac{1 - \lambda_1^2 \lambda_2^2}{\Phi_1 \Phi_2}.
\]

Now let \( f(\lambda_1, \lambda_2) = \lambda_1^a \lambda_2^b \) for arbitrary nonnegative integers \( a \) and \( b \). Then by an application of the binomial theorem we find:

\[
(18) \quad \lambda_1^a \lambda_2^b = \alpha(1 - \alpha) \sum_{i,j} s^{i+j} [\lambda_1^i \lambda_2^j] \lambda_1^a \lambda_2^b (1 - \lambda_1^2 \lambda_2^2) \Phi_1^{i-1} \Phi_2^{j-1}
\]

\[
= \sum_{i \geq (a+b)/2} s^{2i-a-b} \sum_{\ell=0}^{i-1} \alpha^{2i-2\ell-a}(1 - \alpha)^{2\ell+b} \times
\]

\[
\times \left( i - 1 \right) \left[ \binom{i - a + b - 1}{\ell} - \delta^2 \binom{i - a + b - 1}{\ell + b + 1} \right],
\]

with \( \delta = (1 - \alpha)/\alpha \).

Setting \( a = 1 \) and \( b = 0 \) in (18) we obtain the series expansion of \( F \) and have

\[
P(T_1 = 2n + 1) = \alpha^{2n+1} \sum_{\ell \geq 1} \delta^{2\ell} \binom{n}{\ell} \left( \frac{n-1}{\ell-1} \right) - \delta^2 \binom{n-1}{\ell+1}
\]

By splitting the summation this may be simplified further to the neat formula\(^1\)

\[
(19) \quad P(T_1 = 2n + 1) = \alpha^{2n+1} \sum_{\ell \geq 0} \delta^{2\ell+2} \frac{n}{\ell+1} \binom{n}{\ell} \left( \frac{n-1}{\ell} \right).
\]

If we set \( a = 0 \) and \( b = 1 \) in (18) we obtain the series expansion of \( G \), which we may get more directly by replacing \( \alpha \rightarrow 1 - \alpha \) in (19):

\[
(20) \quad P(T_1^* = 2n + 1) = \alpha^{2n+1} \sum_{\ell \geq 0} \delta^{-2\ell-2} \frac{n}{\ell+1} \binom{n}{\ell} \left( \frac{n-1}{\ell} \right).
\]

\(^1\)Here we have used the relation

\[
\binom{n}{\ell+1} \left( \frac{n-1}{\ell} \right) - \binom{n}{\ell} \left( \frac{n-1}{\ell+1} \right) = \frac{1}{\ell+1} \binom{n}{\ell} \left( \frac{n-1}{\ell} \right).
\]
For the series expansion of $(FG)^a$ we obtain

$$P(T_{2a} = 2n) = \alpha^{2n} \sum_{\ell \geq 0} \delta^{2\ell+a} \binom{n-1}{\ell} \left[ \left( \binom{n-1}{\ell+a-1} - \delta^2 \binom{n-1}{\ell+a+1} \right) \right].$$

(21)

An finally for $(FG)^a F$:

$$P(T_{2n+1} = 2a + 1) = \alpha^{2n+1} \sum_{\ell \geq 0} \delta^{2\ell+a} \binom{n}{\ell} \left[ \left( \binom{n-1}{\ell+a-1} - \delta^2 \binom{n-1}{\ell+a+1} \right) \right].$$

(22)

An interesting question turns up at this point: How does $P(T_a = n)$ behave as $n \to \infty$? In the case of a simple symmetric random walk, i.e. $\alpha = 1/2$, it is well known that

$$P(T_a = n) \sim \frac{a\sqrt{2}}{\sqrt{n^3\pi}},$$

(e.g. Feller (1968, p. 90)). It turns out that a similar results holds also for two-periodic walks with an additional modulating factor depending on $\alpha$. We will derive this result directly from the generating functions (12) and (13) by means of singularity analysis, a powerful method developed by Flajolet and Odlyzko (1990). Its basic principle is the correspondence between the asymptotic expansion of a generating function near its dominant singularities and the asymptotic expansion of its coefficients.

A closer examination of (9) exhibits that $s = 0$ is a removable singularity. Moreover, the discriminant in $F$ factors nicely to

$$(1 - s^2(1 - 2\alpha))^2 - 4\alpha^2 s^2 = (1 - s)(1 + s)(1 - s(1 - 2\alpha))(1 + s(1 - 2\alpha)).$$

Hence $F$ has four algebraic singularities (branch points) located at

$$s_{1,2} = \pm 1, \quad \text{and} \quad s_{3,4} = \pm \frac{1}{1 - 2\alpha},$$

and the dominant singularities are $s = 1$ and $s = -1$, since $|s_{3,4}| > 1$. Let us first expand $(FG)^a$ locally around $s = \pm 1$. We obtain

$$\begin{align*}
(FG)^a &= -\sqrt{1 \pm s} \frac{a\sqrt{2}}{\sqrt{\alpha(1 - \alpha)}} + O((1 - s)^{3/2}).
\end{align*}$$

(23)

A trite calculation shows that $G$ has exactly the same discriminant as $F$.\footnote{A trite calculation shows that $G$ has exactly the same discriminant as $F$.}
If we take formally Taylor coefficients in (23), we get
\[
[s^n](FG)^a = [s^n] \left( -\sqrt{1 + s} \frac{a\sqrt{2}}{\sqrt{\alpha(1 - \alpha)}} \right) + [s^n]O((1 - s)^{3/2})
\]
(24)
\[
= [s^n] \left( -\sqrt{1 + s} \frac{a\sqrt{2}}{\sqrt{\alpha(1 - \alpha)}} \right) + O([s^n](1 - s)^{3/2})
\]
The transition from line one to line two in the above equation, i.e.,
\[
[s^n]O((1 - s)^{3/2}) = O([s^n](1 - s)^{3/2})
\]
(25)
looks harmless, but it is in fact nontrivial. This transfer of error terms is guaranteed under conditions of analytic continuation by transfer theorems. For a full elaboration of these theorems see Flajolet and Sedgewick (2008), chapter 6, in particular Theorem 6.3.

The functions \(\sqrt{1 - s}\) and \((1 - s)^{3/2}\) belong to an asymptotic scale of standard functions of type \(\{(1 - s)^{-a}\}_{a \in \mathbb{R}}\) for which full asymptotic expansions are available. In particular (Flajolet and Sedgewick (2008, p. 381)),
\[
[s^n](1 - s)^{-a} \sim \frac{n^{a-1}}{\Gamma(a)} \left[ 1 + \sum_{k=1}^{\infty} \frac{c_k}{n^k} \right]
\]
\[\sim \frac{n^{a-1}}{\Gamma(a)} \left[ 1 + \frac{a(a - 1)}{2n} + \frac{a(a - 1)(a - 2)(3a - 1)}{24n^2} + \frac{a^2(a - 1)^2(a - 2)(a - 3)}{48n^3} + O(n^{-4}) \right].
\]
Therefore it follows from (24), (25) and
\[
[s^n](1 - s)^{-a} = (-1)^n[s^n](1 + s)^{-a},
\]
that
\[
P(T_{2a} = n) = \frac{a(1 + (-1)^n)}{\sqrt{\alpha(1 - \alpha)}\sqrt{2n^3\pi}} + O(n^{-5/2}),
\]
(26)
Applying the same procedure to the generating function \((FG)^a F\), we get
\[
(FG)^a F = -\sqrt{2}(a + 1 - \alpha) \sqrt{1 - s} + O((1 - s)^{3/2}) \quad \text{at } s = 1
\]
\[
= \frac{\sqrt{2}(a + 1 - \alpha)}{\sqrt{\alpha(1 - \alpha)}} \sqrt{1 + s} + O((1 + s)^{3/2}) \quad \text{at } s = -1,
\]
which implies
\[
P(T_{2a+1} = n) = \frac{(a + 1 - \alpha)(1 - (-1)^n)}{\sqrt{\alpha(1 - \alpha)}\sqrt{2n^3\pi}} + O(n^{-5/2}).
\]
(27)
Let us now turn to the maximum of $S_n$, $M_n = \max_{0 \leq i \leq n} S_n$. Since
\begin{equation}
\label{eq:28}
P(M_n \geq a) = P(T_n \leq n), \quad a = 1, 2, \ldots,
\end{equation}
the distribution of $M_n$ follows easily by a summation on (21) and (22). In particular,
\begin{align}
P(M_n \geq 2a) &= \sum_{i=0}^{\lfloor n/2 \rfloor} \alpha^i \sum_{\ell \geq 0} \delta^{2\ell + a} \left( \binom{i-1}{\ell} \right) \left[ \binom{i-1}{\ell + a - 1} - \delta^2 \binom{i-1}{\ell + a + 1} \right], \\
P(M_n \geq 2a + 1) &= \sum_{i=0}^{\lfloor n/2 \rfloor} \alpha^{i+1} \sum_{\ell \geq 0} \delta^{2\ell + a} \left( \binom{i}{\ell} \right) \left[ \binom{i-1}{\ell + a - 1} - \delta^2 \binom{i-1}{\ell + a + 1} \right]
\end{align}

To obtain the asymptotic distribution of $M_n$ we require its pgf., which in light of (28) is given by
\begin{align}
\sum_{n \geq 0} s^n P(M_n \geq 2a) &= \frac{(FG)^a}{1 - s}, \quad \text{and} \quad \sum_{n \geq 0} s^n P(M_n \geq 2a + 1) = \frac{(FG)^a F}{1 - s}.
\end{align}

A full asymptotic expansion is now possible for fixed $a$ following exactly the same way as we pursued above when applying singularity analysis. Here, however, the case $a = x \sqrt{n}$, $x \in \mathbb{R}^+$ is more interesting.

By Cauchy’s formula we have
\begin{equation}
P(M_n \geq 2a) = \frac{1}{2\pi i} \int_C \frac{(FG)^a}{1 - s} \frac{ds}{s^{n+1}},
\end{equation}
where the contour of integration is the circle $s = Re^{\theta i}$, with $-\pi < \theta < \pi$ and $0 < R < 1$. Now put $s = e^{-t/n}$. This substitution maps the contour to the new contour
\begin{equation}
C: t = -n \log R - n \theta i,
\end{equation}
which is a straight line segment in the right half plane, parallel to the imaginary axis and oriented from top to bottom. This substitution leads to
\begin{equation}
P(M_n \geq 2a) = \frac{1}{2\pi i} \int_C \frac{(FG)^a}{1 - e^{-t/n}} \frac{e^t}{n} dt.
\end{equation}

Let us expand first
\begin{equation}
\frac{1}{1 - e^{-t/n}} = \frac{1}{t} + \frac{1}{2n} + \frac{t}{12n^2} + \ldots.
\end{equation}

Expansion of $\ln(FG)^a$ yields when setting $a = x \sqrt{n}$:
\begin{equation}
\ln(FG)^{x \sqrt{n}} = -x \sqrt{\frac{2}{\alpha(1 - \alpha)}} \sqrt{t} + \frac{x \sqrt{2(1 - 6\alpha + 6\alpha^2)}}{12n[\alpha(1 - \alpha)]^{3/2}} t^{3/2} + \ldots
\end{equation}
Changing the orientation of the contour $C$ and thereby changing the sign of the integral we get as $n \to \infty$:

$$P(M_n \geq 2x\sqrt{n}) \to \frac{1}{2\pi i} \int_{-C} \exp \left[ -x \sqrt{\frac{2}{\alpha(1-\alpha)}} \sqrt{t} \right] \frac{e^t}{t} \, dt,$$

which turns out (Erdélyi (1954, p. 245)) to be the complex inversion integral of the Laplace transform of

$$\text{erfc} \left( \frac{x}{\sqrt{2\alpha(1-\alpha)}} \right).$$

In the case $M_n \geq 2a + 1$ we have clearly the same limit. Hence we have proved that

$$(30) \quad P(M_n < x\sqrt{n}) \to 2\Phi \left( \frac{x}{2\sqrt{\alpha(1-\alpha)}} \right) - 1.$$

The limiting distribution is therefore stable with index $1/2$. For $\alpha = 1/2$ we have

$$P(M_n < x\sqrt{n}) \to 2\Phi(x) - 1,$$

which has been found by Rényi (1970, p. 234). It may be remarked that a full asymptotic expansion is possible. The fractional power series (29) has terms containing $t^{n+1/2}$, which for $n > 0$ give rise to the inversion integrals of parabolic cylinder functions.

We conclude this section with a brief discussion of the expected value of $M_n$. Since

$$E(M_n) = \sum_{a \geq 1} P(M_n \geq a) = \sum_{a \geq 1} P(T_a \leq n),$$

the generating function of $E(M_n)$ is given by

$$\Psi(s) = \sum_{n \geq 0} s^n E(M_n) = \frac{1}{1 - s} \frac{F + FG}{1 - FG}.$$

Explicit formulas for $E(M_n)$ are rather difficult to get, but asymptotic formulas are readily available by singularity analysis. Before we carry out this analysis let us first have a look at $E(M_n)$ for small value of $n$, which can be calculated
explicitly e.g. by Maple. We observe an interesting pattern:

\[
\begin{align*}
E(M_1) &= \alpha & \text{max in } \alpha = 1 \\
E(M_2) &= (2 - \alpha)\alpha & \text{max in } \alpha = 1 \\
E(M_3) &= (3 - 2\alpha)\alpha & \text{max in } \alpha = 0.75 \\
E(M_4) &= (4 - 4\alpha + 2\alpha^2 - \alpha^3)\alpha & \text{max in } \alpha = 0.6941 \\
E(M_5) &= (5 - 6\alpha + 4\alpha^2 - 2\alpha^3)\alpha & \text{max in } \alpha = 0.6611 \\
E(M_6) &= (6 - 9\alpha + 10\alpha^2 - 10\alpha^3 + 6\alpha^4 - 2\alpha^5)\alpha & \text{max in } \alpha = 0.6436 \\
E(M_7) &= (7 - 12\alpha + 16\alpha^2 - 18\alpha^3 + 12\alpha^4 - 4\alpha^5)\alpha & \text{max in } \alpha = 0.6297 \\
\end{align*}
\]

These experiments exhibit a somewhat unexpected phenomenon: \( E(M_n) \) attains its maximal value for \( \alpha > 1/2 \). This pattern prevails, as an asymptotic analysis shows. The function \( \Psi(s) \) inherits its singularities from \( F \), the dominant ones are again \( s = \pm 1 \). Expanding \( \Psi(s) \) around \( s = 1 \) we obtain

\[
\Psi(s) = \sqrt{2\alpha(1 - \alpha)}(1 - s)^{-3/2} - (1 - \alpha)(1 - s)^{-1} + \frac{1 - 5\alpha + 5\alpha^2}{2\sqrt{2\alpha(1 - \alpha)}}(1 - s)^{-1/2} + O((1 - s)^{1/2}),
\]

whereas an expansion around \( s = -1 \) shows that \( \Psi(s) = O((1 + s)^{1/2}) \). Hence by the transfer principle and the asymptotic expansions of \( (1 - s)^{k/2} \) we obtain

\[
E(M_n) \sim -1 + \alpha + \sqrt{\frac{2\alpha(1 - \alpha)n}{\pi}} \left[ 2 + \frac{1}{4n} \left( 3 + \frac{1 - 5\alpha + 5\alpha^2}{\alpha(1 - \alpha)} \right) \right] + O(n^{-3/2}).
\]
Dropping terms of order $n^{-1/2}$ and higher, we find that $E(M_n)$ has its maximum approximately at

$$\tilde{\alpha} = \frac{1}{2} + \frac{1}{2} \sqrt{\frac{\pi}{\pi + 8n}}$$

When $n \to \infty$, then clearly $\tilde{\alpha} \to \frac{1}{2}$.

3. Transition probabilities and returns to zero

Let $p_n(k) = P(S_n = k)$ and $q_n(k) = P(S_n^* = k)$ with corresponding pgf.s

$$P_k := P_k(s) = \sum_{n \geq 0} s^np_n(k), \quad \text{and} \quad Q_k := Q_k(s) = \sum_{n \geq 0} s^nq_n(k).$$

Explicit formulas for $p_n(k)$ and $q_n(k)$ can be derived by a rather simple combinatorial argument\(^3\). We demonstrate the procedure here only for $p_{2n}(2k)$. Our emphasis in this section will be put on the generating functions, since they are what we need in section 4, when we are discussing two-periodic walks with boundaries.

The event $\{S_{2n} = 2k\}$ requires that the corresponding paths of $S_n$ consist of

- $o^+$ up-steps at odd height
- $e^+$ up-steps at even height
- $o^−$ down-steps at odd height
- $e^−$ down-steps at even height

These numbers have to satisfy

$$o^+ + o^− + e^+ + e^− = 2n$$
$$o^+ − o^− + e^+ − e^− = 2k$$
$$o^+ + o^− − e^+ − e^− = 0$$

Putting $o^+ = \ell$ and solving this system of equations yields

$$o^+ = \ell, \quad o^− = n − \ell, \quad e^+ = n + k − \ell, \quad e^− = \ell − k.$$

\(^3\)Not really a tiresome combinatorial task, as Conolly et al. (1997) remarked.
Hence
\[ p_{2n}(2k) = \sum_{\ell \geq k} \binom{n}{\ell} \binom{n}{\ell - k} \alpha^{2n-2\ell+k}(1 - \alpha)^{2\ell-k} \]
\[ = \sum_{\ell \geq 0} \binom{n}{\ell} \binom{n}{\ell + k} \alpha^{2n-2\ell-k}(1 - \alpha)^{2\ell+k} \]
\[ = \alpha^{2n} \sum_{\ell} \binom{n}{\ell} \binom{n}{\ell + k} \delta^{2\ell+k}. \]  
(31)

In a similar manner \( p_{2n+1}(2k + 1) \) and \( q_{2n+1}(2k + 1) \) may be determined. Note that \( p_{2n}(2k) = q_{2n}(2k) \).

In the sequel we will derive some important functional relations between the pgf.s \( P_k \) and \( Q_k \) and show how they are related to the pgf.s \( F \) and \( G \) of the first passage times \( T_1 \) and \( T_1^* \).

Consider first \( P_{2k} \). A path terminating after \( 2n \) steps at altitude \( 2k \), \( k \in \mathbb{Z} \) must reach altitude \( 2k \) for a first time and may then return to that altitude an arbitrary number of times. Thus
\[ (32) \quad P_{2k} = (FG)^kP_0 = Q_{2k} \quad \text{for all } k \in \mathbb{Z}. \]

The same argument shows that
\[ (33) \quad P_{2k+1} = (FG)^kFP_0 \]
\[ P_{-2k-1} = (FG)^kGP_0 \quad \text{for } k \geq 0. \]

Furthermore by quasi symmetry of \( S_n \) and \( S_n^* \) we have
\[ (34) \quad Q_{2k+1} = P_{-2k-1} \quad \text{and} \quad Q_{-2k-1} = P_{2k+1} \quad \text{for } k \geq 0. \]

It remains to find \( P_0 \). Let \( \tau_0 \) denote the epoch of the first return to zero of \( S_n \) with pgf. \( \Upsilon(s) = \sum_{n \geq 0} s^n P(\tau_0 = n) \). Conditioning on the first step we have
\[ \Upsilon = \alpha sF + (1 - \alpha)sG \]
\[ = 1 - \sqrt{(1 - s^2(1 - 2\alpha))^2 - 4\alpha^2 s^2}. \]  
(35)

But \( P_0 = \sum_{k \geq 0} \Upsilon^k \), thus
\[ (36) \quad P_0 = \frac{1}{1 - \Upsilon} = \frac{1}{\sqrt{(1 - s^2(1 - 2\alpha))^2 - 4\alpha^2 s^2}}. \]

Consider now generating functions of the form
\[ (37) \quad W_{a,b}(s) := W_{a,b}(s) = \sum_{n \geq 0} s^n w_{a,b}^n(n) = F^n G^b P_0, \]
for nonnegative integers $a$ and $b$. Series expansions of these functions are most conveniently obtained again by multivariate Lagrange inversion. Putting $\lambda_1 = F, \lambda_2 = G$ the set of functional equations relating $F$ and $G$ becomes

$$
\Phi_1 = \Phi_1(\lambda_1, \lambda_2) = \alpha + (1 - \alpha)\lambda_1\lambda_2
$$

$$
\Phi_2 = \Phi_2(\lambda_1, \lambda_2) = 1 - \alpha + \alpha\lambda_1\lambda_2
$$

with

$$
w_1 = s\Phi_1(w_1, w_2)
$$

$$
w_2 = s\Phi_2(w_1, w_2),
$$

$w_1$ and $w_2$ being arbitrary indeterminates. Let $f(\lambda_1, \lambda_2)$ be a formal Laurent series, then by the multivariate Lagrange inversion formula:

$$
f(w_1(s), w_2(s)) = \sum_{i,j} \left[ \lambda_1^i \lambda_2^j \right] f(\lambda_1, \lambda_2) \Phi_1^i \Phi_2^j \det \left\| \delta_{xy} - \frac{\lambda_y}{\Phi_x} \frac{\partial \Phi_x}{\partial \lambda_y} \right\|, \quad x, y = 1, 2,
$$

Now observe that the pgf.s $W_{a,b}$ translate to

$$
f(\lambda_1, \lambda_2) = \frac{\lambda_1^a \lambda_2^b}{1 - \alpha s\lambda_1 - (1 - \alpha) s\lambda_2}.
$$

But by (38)

$$
s = \frac{\lambda_1}{\Phi_1} = \frac{\lambda_2}{\Phi_2}.
$$

So

$$
1 - \alpha s\lambda_1 - (1 - \alpha) s\lambda_2 = 1 - \frac{(1 - \alpha)\lambda_1\lambda_2}{\Phi_1} - \frac{\alpha\lambda_1\lambda_2}{\Phi_2} = \det \left\| \delta_{xy} - \frac{\lambda_y}{\Phi_x} \frac{\partial \Phi_x}{\partial \lambda_y} \right\|.
$$

It follows that the Jacobian in (39) cancels and the $W_{a,b}$ take on the particularly simple form:

$$
W_{a,b}(s) = \sum_{i,j} s^{i+j} \left[ \lambda_1^i \lambda_2^j \right] \lambda_1^a \lambda_2^b \Phi_1^i \Phi_2^j
$$

$$
= \sum_{i \geq a, j \geq b} s^{i+j} \left[ \lambda_1^{i-a} \lambda_2^{j-b} \right] (\alpha + (1 - \alpha)\lambda_1\lambda_2)^i (1 - \alpha + \alpha\lambda_1\lambda_2)^j
$$

$$
= \sum_{i \geq a} s^{2i-a+b} \sum_{\ell=0}^{i-a} \binom{i}{\ell} \binom{i-a+b}{\ell+b} \alpha^{2i-a-2\ell}(1 - \alpha)^{2\ell+b}
$$

$$
= \sum_{i \geq a} s^{2i-a+b} \alpha^{2i-a+b} \sum_{\ell=0}^{i-a} \delta^{2\ell+b} \binom{i}{\ell} \binom{i-a+b}{\ell+b}.
$$

(40)
Thus the functions $w^b_a(n)$ can be read off directly from (40):

\[
w^b_a(n) = \alpha^n \sum_{\ell=0}^{(n-a-b)/2} \delta^{2\ell+b} \left( \frac{a+b}{2} \ell \right) \left( \frac{n}{2} \ell + b \right).
\]

We remark in passing that in (40) $a$ and/or $b$ may also be negative integers. In this case (40) yields a Laurent series in $s$, which is not what we want, since $W_{a,b}$ must be a power series. Still it can be shown that the non-principal part of these Laurent series coincides term by term with $W_{a,b}$.

Let us now list some particular cases of the functions $w^b_a(n)$. Firstly, a simple calculation shows that the following symmetry relation holds:

\[
w^b_a(n) = w^{-b}_{-a}(n).
\]

For $a = b = k$ we obtain

\[W_{k,k} = P_{2k} = Q_{2k} = \sum_{i \geq k} s^{2i} \sum_{\ell \geq 0} \binom{i}{\ell} \binom{i}{\ell + k} \alpha^{2i-k-2\ell}(1-\alpha)^{2\ell+k},\]

It follows that

\[p_{2n}(2k) = q_{2n}(2k) = w^k_k(2n) = \alpha^{2n} \sum_{\ell \geq 0} \delta^{2\ell+k} \binom{n}{\ell} \binom{n}{\ell + k}.
\]

For $a = k + 1$, $b = k$ and $k \geq 0$:

\[W_{k+1,k} = P_{2k+1} = Q_{-2k-1} = \sum_{i \geq k+1} s^{2i-1} \sum_{\ell \geq 0} \binom{i-1}{\ell} \binom{i}{\ell + k} \alpha^{2i-2\ell-k-1}(1-\alpha)^{2\ell+k},\]

which implies

\[p_{2n+1}(2k+1) = q_{2n+1}(-2k-1) = w^k_{k+1}(2n+1) = \alpha^{2n+1} \sum_{\ell \geq 0} \delta^{2\ell+k} \binom{n+1}{\ell} \binom{n}{\ell + k}.
\]

And finally for $a = k$ and $b = k + 1$:

\[W_{k,k+1} = Q_{2k+1} = P_{-2k-1} = \sum_{i \geq k} s^{2i+1} \sum_{\ell \geq 0} \binom{i+1}{\ell} \binom{i}{\ell + k + 1} \alpha^{2i-k-2\ell}(1-\alpha)^{2\ell+k+1},\]
which yields
\[
q_{2n+1}(2k+1) = p_{2n+1}(-2k-1) = w_{k+1}^b(2n+1)
\]

\[
= \alpha^{2n+1} \sum_{\ell \geq 0} \delta^{2\ell+k+1} \left( \frac{n}{\ell} \right) \left( \frac{n + 1}{\ell + k + 1} \right).
\]

Later, in section 4, we will find that the functions \( w_a^b(n) \) (not only those special cases listed above) play a prominent role in the derivation of the distribution of \( S_n \) in case of one or two absorbing boundaries. For those purposes it will be quite convenient to have also an asymptotic approximation of these functions. Such an approximation can be derived as follows.

By Bragg (1999) \( w_a^b(n) \) can be represented as a Hadamard product of the series

\[
C(\xi) = \sum_{\ell \geq -b} \xi^\ell \left( \frac{n+a-b}{2} \right) = (1 + \xi)^{(n+a-b)/2} \quad \text{and}
\]

\[
D(\zeta) = \zeta^{-b} \sum_{\ell \geq 0} \zeta^\ell \left( \frac{n-a+b}{2} \right) = \zeta^{-b}(1 + \zeta)^{(n-a+b)/2}.
\]

Setting \( \xi = \delta e^{bi} \) and \( \zeta = \delta e^{-bi} \) we find that the functions \( w_a^b(n) \) have the integral representation:

\[
w_a^b(n) = \frac{\alpha^n}{2\pi\sqrt{n}} \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} \left( 1 + \delta e^{bi/\sqrt{n}} \right)^{n+a-b} \left( 1 + \delta e^{-bi/\sqrt{n}} \right)^{n-a+b} e^{b\theta i} d\theta.
\]

Let us now put \( \theta = t/\sqrt{n} \) and \( a = u\sqrt{n}, b = v\sqrt{n} \):

\[
w_a^b(n) = \frac{\alpha^n}{2\pi\sqrt{n}} \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} \left( 1 + \delta e^{ti/\sqrt{n}} \right)^{n+\sqrt{n}(u-v)} \left( 1 + \delta e^{-ti/\sqrt{n}} \right)^{n-\sqrt{n}(u-v)} e^{vti} dt
\]

Around \( t = 0 \) we expand the log of \( \left( 1 + \delta e^{ti/\sqrt{n}} \right)^{n+\sqrt{n}(u-v)} \left( 1 + \delta e^{-ti/\sqrt{n}} \right)^{n-\sqrt{n}(u-v)} \) to obtain

\[
\ln \left( \left( 1 + \delta e^{ti/\sqrt{n}} \right)^{n+\sqrt{n}(u-v)} \left( 1 + \delta e^{-ti/\sqrt{n}} \right)^{n-\sqrt{n}(u-v)} \right) = \sum_{k \geq 0} c_k(t)^k
\]
For the coefficients $c_k$ we find:

\[
\begin{align*}
    c_0 &= \ln(1 + \delta)n = \ln \alpha^{-n} \\
    c_1 &= (1 - \alpha)(u - v) \\
    c_2 &= -\frac{\alpha(1 - \alpha)}{2} \\
    c_3 &= \frac{\alpha(1 - \alpha)(1 - 2\alpha)(u - v)}{6n} \\
    c_4 &= \frac{\alpha(1 - \alpha)(1 - 6\alpha + 6\alpha^2)}{24n} \\
    c_5 &= O(1/n^2)
\end{align*}
\]

Thus

\[
    w_0^b(n) \sim \frac{1}{2\pi \sqrt{n}} \int_{-\infty}^{\infty} e^{-\frac{\alpha(1-\alpha)}{2}t^2} e^{it(\alpha v + (1-\alpha)u)} \left[ 1 + ic_3 t + c_4 t^2 \right] dt.
\]

An evaluation these integrals with $a = u\sqrt{n}, b = v\sqrt{n}$ and $\xi = \alpha v + (1 - \alpha)u$ results in the approximation formula:

\[
    w_0^b(n) \sim \frac{1}{\sqrt{2\pi n}} \frac{1}{\sqrt{\alpha(1 - \alpha)}} \exp \left[ -\frac{\xi^2}{2\alpha(1 - \alpha)} \right] \times
\]

\[
    \times \left[ 1 + \frac{(u - v)\xi(1 - 2\alpha)[\xi^2 - 3\alpha(1 - \alpha)]}{6\alpha^2(1 - \alpha)^2} + \frac{(1 - 6\alpha + 6\alpha^2)[\xi^4 - 6\alpha(1 - \alpha)\xi^2 + 3\alpha^2(1 - \alpha)^2]}{24\alpha^3(1 - \alpha)^3} \right].
\]

This approximation provides remarkable accuracy, as can be seen from the following example: with $a = 0, b = 0, \alpha = 0.3$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$w_0^b(n)$ exact</th>
<th>$w_0^b(n)$ approx.</th>
<th>error in %</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.27156415</td>
<td>0.27103579</td>
<td>0.19456</td>
</tr>
<tr>
<td>20</td>
<td>0.19323610</td>
<td>0.19315757</td>
<td>0.04064</td>
</tr>
<tr>
<td>30</td>
<td>0.15814977</td>
<td>0.15812247</td>
<td>0.01727</td>
</tr>
<tr>
<td>40</td>
<td>0.13712864</td>
<td>0.13711560</td>
<td>0.00951</td>
</tr>
<tr>
<td>50</td>
<td>0.12274257</td>
<td>0.12273519</td>
<td>0.00601</td>
</tr>
</tbody>
</table>

Thus (46) gives even the value of $w_0^b(10)$ with an error of less than 0.2 %.

4. Absorbing Boundaries

In this section we assume that the motion of $S_n$ is restricted by one or two absorbing boundaries, an upper boundary located at $U$, a lower boundary at
$-L, U$ and $L$ being positive integers. We are interested in finding the following probabilities:

\[
p_{n,k}^{(U)} = P(S_n = k, S_i < U, i = 0, 1, \ldots, n|S_0 = 0)
\]

\[
p_{n,k}^{(-L)} = P(S_n = k, S_i > -L, i = 0, 1, \ldots, n|S_0 = 0)
\]

\[
p_{n,k}^{(U,-L)} = P(S_n = k, -L < S_i < U, i = 0, 1, \ldots, n|S_0 = 0),
\]

with corresponding pgf.s $P_k^{(U)}, P_k^{(-L)}$ and $P_k^{(U,-L)}$.

Our analysis is based on the observation that the pgf.s $P_k$ and $Q_k$ together with the pgf.s $F$ and $G$ are also path generating functions in the sense that they give us the weight of sets of lattice paths representing the sample paths of $S_n$ having certain well-defined properties. Such paths may be decomposed in particular ways and these decompositions translate in a simple and transparent manner into algebraic manipulations of the generating functions. When dealing with absorbing boundaries a quite natural decomposition is induced by the epoch when a path touches a boundary for the first time. By the two-periodic character of $S_n$ we expect that the parity of the boundaries will play an important role.

Let us start with the case of one upper boundary located at $U = 2a$ and consider first paths with terminal altitude $2k, k < a$. Any path which touches the boundary must do so for a first time. This initial path segment has pgf. $(FG)^a$. It is followed by a path segment which starts at even altitude $2a$ and terminates at altitude $2k$. Thus we have

\[
P_{2k}^{(2a)} = P_{2k} - (FG)^a P_{2k-2a}
\]

Now $P_{2k-2a} = P_{2a-2k} = (FG)^{a-k} P_0$, therefore we have

\[
P_{2k}^{(2a)} = P_{2k} - (FG)^{2a-k} P_0 = W_{k,k} - W_{2a-k;2a-k}
\]

\[
= P_{2k} - P_{4a-2k}
\]

It follows that

\[
p_{2n,2k}^{(2a)} = w_{k}^{(2n)} - w_{2a-k}^{(2n)}
\]

\[
= p_{2n,2k} - p_{2n,4a-2k} \quad \text{by (43)}.
\]

Observe that this formula is structurally equivalent to the well-known formula we obtain for the simple random walk by the classical reflection principle. The same is true when paths terminate at altitude $2k + 1$. Now the decomposition at the first-passage point yields

\[
P_{2k+1}^{(2a)} = P_{2k+1} - (FG)^a P_{2k+1-2a}
\]
Since \( 2k + 1 < 2a \) we have
\[
(FG)^a P_{2k+1-2a} = (FG)^a Q_{2a-2k-1}
= (FG)^a (FG)^{a-k-1} GP_0
= (FG)^{2a-k-1} GP_0
= W_{2a-k-1,2a-k}
\]
Therefore
\[
p_{2n+1,2k+1}^{(2a)} = w_k^{(2n+1)}(2n + 1) - w_{2a-k-1}^{(2n+1)}(2n + 1)
= p_{2n+1,2k+1} - p_{2n+1,2k-4a+1} \quad \text{by (44).}
\]
Suppose now that \( U = 2a + 1 \). If paths terminate at altitude \( 2k \), then the structural similarity to the formulas of the simple random walk obtained by the reflection principle no longer holds. Paths which touch the boundary do so with a first passage through \( 2a + 1 \), the corresponding path segment has pgf. \( (FG)^a F \), but the trailing path segment now starts at odd altitude \( 2a + 1 \) and leads to altitude \( 2k \). It has pgf. \( Q_{2k-2a-1} \). Hence
\[
P_k^{(2a+1)}(2k) = P_{2k} - (FG)^a F Q_{2k-2a-1}
= P_{2k} - (FG)^a F P_{2(a-k)+1}
= P_{2k} - (FG)^a F (FG)^{a-k} F P_0
= P_{2k} - P_{2a-k+2} G^{2a-k} P_0
= W_{k,k} - W_{2a-k+2,2a-k},
\]
and this cannot be expressed directly in terms of the functions \( P_k \). We have in this case
\[
p_{2n,2k}^{(2a+1)} = w_k^{(2n)} - w_{2a-k+2}(2n).
\]
And finally:
\[
P_{2k+1}^{(2a+1)} = P_{2k+1} - (FG)^a F Q_{2k-2a}
= P_{2k+1} - (FG)^a F P_{2a-2k}
= P_{2k+1} - (FG)^{2a-k} F P_0
= W_{k+1,k} - W_{2a-k+1,2a-k}
= P_{2k+1} - P_{4a-2k+1},
\]
which yields
\[
p_{2n+1,2k+1}^{(2a+1)} = w_k^{(2n+1)}(2n + 1) - w_{2a-k+1}(2n + 1)
= p_{2n+1,2k+1} - p_{2n+1,4a-2k+1}.
\]
If we have instead a lower boundary at \(-L\), then the steps of proof are exactly as above, so we omit the details here and summarize the one-boundary case in the
following theorem which covers all possible cases and all results are expressed in
terms of the basic functions $w^b_0(n)$:

**Theorem 4.1** (One absorbing boundary). For $i = 0, 1$:

$$
p_{2n+1,2k+i}^{(2a)} = w_{k+i}^b(2n+i) - w_{2a-k}^{2a-k-1}(2n+i)$$

$$
p_{2n+1,2k+i}^{(2a+1)} = w_{k+i}^b(2n+i) - w_{2a-k+2}^{2a-k+1}(2n+i)$$

$$
p_{2n+1,2k+i}^{(-2a)} = w_{k+i}^b(2n+i) - w_{2a+k}^{2a+k+1}(2n+i)$$

$$
p_{2n+1,2k+i}^{(-2a-1)} = w_{k+i}^b(2n+i) - w_{2a+k+2}^{2a+k+1}(2n+i).$$

Suppose now that there are two absorbing boundaries, one located at $U$ and one at $-L$, and we assume that $U$ and $L$ are positive integers. Let

$$p_{U-L}^{(n,k)} = P(S_n = k, -L < S_i < U, i = 0, 1, \ldots, n|S_0 = 0).$$

Formulas for these transition functions may derived using Theorem 4.1 and the principle of inclusion-exclusion (see Mohanty (1979, pp. 6)). The latter leads to following decomposition: from the set of all paths of $S_n$ terminating in $k$ we first subtract those paths which touch or cross $U$ and $L$. Then we add paths which touch or cross first $U$ and then $L$ plus those paths which first reach $L$ and then $U$ and so on. Symbolically:

$$
\{\text{all paths}\} - \{U\} - \{L\} + \{UL\} + \{LU\} - \{ULU\} - \{LUL\} \ldots
$$

Clearly the parity of $U$, $L$ and $k$ will again play an important role, so a case-by-case analysis is required. The results of this analysis are given in the following theorem:

**Theorem 4.2** (Two absorbing boundaries). Let $\beta = 2(a + b)$, then for $i = 0, 1$:

$$
p_{2n+1,2k+i}^{(2a-2b)} = w_{k+i}^b(2n+i) - \sum_{\ell \geq 0} \left[ w_{\beta\ell+2a-k}^{\beta\ell+2a-k-1}(2n+i) - w_{\beta\ell+2a-k-1}^{\beta(\ell+1)-k}(2n+i) \right] - \sum_{\ell \geq 0} \left[ w_{\beta\ell+2b+k}^{\beta\ell+2b+k+1}(2n+i) - w_{\beta(\ell+1)+k}^{\beta(\ell+1)+k+1}(2n+i) \right],$$

$$
p_{2n+1,2k+i}^{(2a+b-2b)} = w_{k+i}^b(2n+i) - \sum_{\ell \geq 0} \left[ w_{\beta\ell+2a-k}^{\beta\ell+2a-k-1}(2n+i) - w_{\beta(\ell+1)-k}^{\beta(\ell+1)-k-1}(2n+i) \right] - \sum_{\ell \geq 0} \left[ w_{\beta\ell+2b+k}^{\beta\ell+2b+k+1}(2n+i) - w_{\beta(\ell+1)+2\ell+k+i}^{\beta(\ell+1)+2\ell+k+i}(2n+i) \right].$$
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\[ p_{2n+i,2k+i}^{(2a,-2b-1)} = w_{k+i}(2n+i) - \sum_{\ell \geq 0} w_{\beta \ell + 2\ell + 2a - k}(2n+i) - w_{\beta \ell (\ell+1)+2\ell - k+2}(2n+i) - \sum_{\ell \geq 0} w_{\beta \ell (\ell+1)+2\ell + 2b + k+i}(2n+i) - \sum_{\ell \geq 0} w_{\beta \ell (\ell+1)+2\ell + 2b + k+i}(2n+i). \]

By the decomposition (52) we have in terms of generating functions:

**Proof.** By the decomposition (52) we have in terms of generating functions:

\[ p_{2k}^{(2a,-2b)} = P_{2k} - (FG)^{a}P_{2k-2a} + (FG)^{b}P_{2k+2b} + \]

\[ + (FG)^{2a+b}P_{2k+2b} + (FG)^{a+2b}P_{2k-2a} - (FG)^{3a+2b}P_{2k-2a} + (FG)^{2a+3b}P_{2k+2b} + \ldots \]

\[ = P_{2k} - P_{2a-2k} \sum_{\ell \geq 0} [(FG)^{a+\beta \ell} - (FG)^{a+2b+\beta \ell}] - P_{2b+2k} \sum_{\ell \geq 0} [(FG)^{b+\beta \ell} - (FG)^{2a+b+\beta \ell}]. \]

Since \( P_{2k-2a} = P_{2a-2k} = (FG)^{a-k}P_{0} = W_{a-k,a-k} \) and \( P_{2b+2k} = (FG)^{b+k}P_{0} = W_{b+k,b+k} \): 

\[ p_{2k}^{(2a,-2b)} = W_{k,k} - \sum_{\ell \geq 0} [W_{\beta \ell + 2a - k, \beta \ell + 2a - k} - W_{\beta (\ell+1) - k, \beta (\ell+1) - k}] - \]

\[ - \sum_{\ell \geq 0} [W_{\beta \ell + 2b + k, \beta \ell + 2b + k} - W_{\beta (\ell+1) + 2b + k, \beta (\ell+1) + 2b + k}]. \]

This proves (53) for \( i = 0 \). Note that the last line may also be written as 

\[ P_{2k}^{(2a,-2b)} = P_{2k} - \sum_{\ell \geq 0} [P_{2\beta \ell + 4a - 2k} - P_{2(\ell+1) - 2k}] - \sum_{\ell \geq 0} [P_{2\beta \ell + 4b + 2k} - P_{2(\ell+1) + 2b + 2k}]. \]

Thus the functions \( p_{2n+i,2k+i}^{(2a,-2b)} \) may also be expressed in terms of the transition probabilities of the unrestricted process \( S_{n} \), an observation we already made in the case of one boundary only. However, when there are two boundaries, this is an exception.
(53) in the case \(i = 1\) is proved exactly in the same way, the only thing that changes is to replace \(P_{2a-2k}\) and \(P_{2b+2k}\) in front of the summations by \(P_{2k+1-2a}\) and \(P_{2b+2k+1}\). These are then resolved in terms of the functions \(W_{i,j}\) by means of relation (37).

When \(U = 2a + 1\) and \(L = 2b\), then decomposition (52) leads to the representation
\[
P_{2k}^{(2a+1, -2b)} = P_{2k} - Q_{2k-2a-1} \sum_{\ell \geq 0} \left[ F^{2\ell+1} (FG)^{\beta+\alpha} - F^{2\ell+1} (FG)^{\beta+\alpha+2b} \right] -
- P_{2k+2b} \sum_{\ell \geq 0} \left[ F^{2\ell} (FG)^{\beta+\alpha+b} - F^{2\ell} (FG)^{\beta+2\alpha+b} \right].
\]

Since \(Q_{2k-1-2a} = P_{2a+1-2k} = (FG)^{a-k}FP_0\) and \(P_{2k+2b} = (FG)^{b+k}P_0\) we get upon resolution in terms of \(W_{i,j}\):
\[
P_{2k}^{(2a+1, -2b)} = W_{k,k} - \sum_{\ell \geq 0} \left[ W_{\beta+2\ell+2a-k, \beta+2\ell+2a-k} - W_{\beta+2\ell+1, \beta+2\ell+1} \right] -
- \sum_{\ell \geq 0} \left[ W_{\beta+2\ell+2b+k, \beta+2\ell+2b+k} - W_{\beta+2\ell+2b, \beta+2\ell+2b} \right],
\]
which proves (54) when \(i = 0\). For \(i = 1\) the proof is almost exactly the same. Note that now it is not possible to express \(P_{2k}^{(2a+1, -2b)}\) in terms of \(P_k\).

When the boundaries are at \(L = -2b - 1\) and \(U = 2a\), then we have:
\[
P_{2k}^{(2a, -2b-1)} = P_{2k} - P_{2k-2a} \sum_{\ell \geq 0} \left[ (FG)^{\beta+\alpha}F^{2\ell} - (FG)^{\beta+\alpha+2b}F^{2\ell+2} \right] -
- Q_{2k+1+2b} \sum_{\ell \geq 0} \left[ (FG)^{\beta+\alpha+b}F^{2\ell+1} - (FG)^{\beta+2\alpha+b}F^{2\ell+1} \right].
\]

In terms of the functions \(W_{ij}\):
\[
P_{2k}^{(2a, -2b-1)} = W_{k,k} - \sum_{\ell \geq 0} \left[ W_{\beta+2a-k, \beta+2\ell+2a-k} - W_{\beta+2\ell+1, \beta+2\ell+1} + 2\ell - k + 2 \right] -
- \sum_{\ell \geq 0} \left[ W_{\beta+2b+k, \beta+2\ell+2b+k} - W_{\beta+2\ell+1, \beta+2\ell+1} + 2\ell + k + 2 \right].
\]
This proves (55) for \(i = 0\) and a similar argument covers the case \(i = 1\). Finally, when \(L = -2b - 1\) and \(U = 2a + 1\), then by (52):
\[
P_{2k}^{(2a+1, -2b-1)} = P_{2k} - Q_{2k-2a-1} \sum_{\ell \geq 0} \left[ (FG)^{\beta+2}F^{\ell+\alpha} + (FG)^{\beta+2}F^{\ell+\alpha+2b+1} \right] -
- Q_{2k+1+2b} \sum_{\ell \geq 0} \left[ (FG)^{\beta+2}F^{\ell+b} - (FG)^{\beta+2}F^{\ell+2a+b+1} \right].
\]
Again in terms of the functions $W_{ij}$:

\[
P_{2k}^{(2a+1, -2b-1)} = W_{k,k} - \sum_{\ell \geq 0} \left[ W_{(\beta+2)\ell+2a-k+2, (\beta+2)\ell+2a-k} - W_{(\beta+2)\ell+\beta-k+2, (\beta+2)\ell+2a-k} \right] - \sum_{\ell \geq 0} \left[ W_{(\beta+2)\ell+2b+k+2, (\beta+2)\ell+2b+k+2} - W_{(\beta+2)\ell+\beta+k+2, (\beta+2)\ell+\beta+k+2} \right].
\]

This proves (56) for $i = 0$ and with an obvious modification also for $i = 1$. □

5. An almost fair coin tossing game

It is well known that the simple random walk in presence of two absorbing boundaries gives rise to the classical *gambler’s ruin problem*. Two players, say Peter and Paul⁴ are tossing in turn an unbiased coin. If Peter starts with an initial capital of $U$ $\$ and Paul has initially $L$ $\$, then the following questions immediately turn up: what is the probability of ultimate ruin of Peter? Is it a fair game? What is the expected duration of the game? In the formulation of Feller the rules are simple: if the coin shows head then Peters wins one dollar from Paul, in case of tail, Peter loses one dollar. There is no rule stating who is tossing the coin, it may be Peter and Paul in turn, but it may also be any third person or even some sophisticated machine. An analysis of this game shows that it is indeed a fair game, i.e., the expected gain of any player is zero, if the coin is unbiased and this fact is independent of the amount of initial capital each player has at his disposal.

Let us now assume that Peter and Paul have only one coin and they know that it is biased. Tail turns up with probability $\alpha \neq 1/2$ and head with probability $1 - \alpha$. Is it still possible for them to have a fair game in this situation?

It is, if rules are slightly changed: Peter and Paul throw the coin in turn with Peter beginning. Now they agree that whenever tail turns up the player loses one $\$ to his adversary. Interestingly, even under these circumstances a fair game is possible, but only in a special situation: the initial amount of capital must be an even number for each player, and in that case the expected gain is zero, no matter how much capital is available initially.

To see this we have to determine the probability of ultimate ruin of Peter. In the framework of a two-periodic walk this is the probability that $S_n$ gets absorbed at $U$ before it touches the boundary $-L$. Let $p_m$ denote this probability provided

---

⁴We continue the tradition introduced by William Feller (Feller (1968, p. 346)) who has given them these names.
$S_0 = m$. The probabilities $p_m$ satisfy the following coupled system of difference equations:

\[
\begin{align*}
p_{2m} &= \alpha p_{2m+1} + (1 - \alpha)p_{2m-1} \\
p_{2m+1} &= (1 - \alpha)p_{2m+2} + \alpha p_{2m},
\end{align*}
\]

with boundary conditions depending on $U$ and $L$. Clearly, $p_0$ is the probability of Peter’s ruin. Put $p_{2m} = x_m$ and $p_{2m+1} = y_m$. Then we have

\[
\begin{align*}
x_m &= \alpha y_m + (1 - \alpha)y_{m-1} \\
y_m &= (1 - \alpha)x_{m+1} + \alpha x_m.
\end{align*}
\]

If we substitute the first equation into the second *et vice versa*, then this system reduces to

\[
\begin{align*}
x_{m+1} - 2x_m + x_{m-1} &= 0 \\
y_{m+1} - 2y_m + y_{m-1} &= 0,
\end{align*}
\]

with general solution

\[
\begin{align*}
p_{2m} &= x_m = c_0 + mc_1 \\
p_{2m+1} &= y_m = d_0 + md_1.
\end{align*}
\]

The constants are determined by the boundary conditions. The following four cases have to be taken care of according to the parity of $L$ and $U$:

**Case 1:** $U = 2a, L = -2b$:

We have $x_a = 1, x_{-b} = 0$ and

\[
y_{a-1} = 1 - \alpha + \alpha x_{a-1}, \quad y_{-b} = (1 - \alpha)x_{-b+1}.
\]

Solving for the constants yields

\[
p_{2m} = \frac{b + m}{a + b}, \quad p_{2m+1} = \frac{b + 1 - \alpha + m}{a + b} \quad \Rightarrow \quad p_0 = \frac{b}{a + b}.
\]

**Case 2:** $U = 2a + 1, L = -2b$:

Now we have $y_a = 1, x_{-b} = 0$ and

\[
x_a = \alpha + (1 - \alpha)y_{a-1}, \quad y_{-b} = (1 - \alpha)x_{-b+1}.
\]

Thus

\[
p_{2m} = \frac{b + m}{a + b + 1 - \alpha}, \quad p_{2m+1} = \frac{b + 1 - \alpha + m}{a + b + 1 - \alpha} \quad \Rightarrow \quad p_0 = \frac{b}{a + b + 1 - \alpha}.
\]

**Case 3:** $U = 2a, L = -2b + 1$:

We have $x_a = 1, y_{-b} = 0$ and

\[
y_{a-1} = 1 - \alpha + \alpha x_{a-1}, \quad x_{-b+1} = \alpha y_{-b+1}.
\]
This yields
\[ p_{2m} = \frac{b + m - 1 + \alpha}{a + b - 1 + \alpha}, \quad p_{2m+1} = \frac{b + m}{a + b - 1 + \alpha} \implies p_0 = \frac{b - 1 + \alpha}{a + b - 1 + \alpha}. \]

**Case 4:** \( U = 2a + 1, L = -2b + 1:\)

In this last case we have \( y_a = 1, y_{-b} = 0 \) and
\[ x_a = \alpha + (1 - \alpha)y_{a-1}, \quad x_{-b+1} = \alpha y_{-b+1}, \]
which yields
\[ p_{2m} = \frac{b + m - 1 + \alpha}{a + b}, \quad p_{2m+1} = \frac{b + m}{a + b} \implies p_0 = \frac{b - 1 + \alpha}{a + b}. \]

Consider now the expected gain of Peter, say \( \gamma \). It is given by
\[ \gamma = -L(1 - p_0) - p_0 U = p_0(L - U) - L. \]

Using the appropriate formula for \( p_0 \), we obtain
\[ \gamma = \begin{cases} 0 & U = 2a, \quad L = -2b & \text{case 1} \\ \frac{(1 - 2\alpha)b}{a + b + 1 - \alpha} & U = 2a + 1, \quad L = -2b & \text{case 2} \\ \frac{(1 - 2\alpha)a}{a + b - 1 + \alpha} & U = 2a, \quad L = -2b + 1 & \text{case 3} \\ 1 - 2\alpha & U = 2a + 1, \quad L = -2b + 1 & \text{case 4} \end{cases} \]

An inspection of this table shows that only case 1 leads to a fair game, in all other cases the game is fair only if \( \alpha = 1/2 \).

Let us have a look also at \( D_0 \), the expected duration of the game. \( D_m \) is a particular value of \( D_m \), the expected time until absorption of \( S_n \) at any of the boundaries, provided \( S_0 = m \). We may argue very similar as we did in the derivation of \( p_m \). The \( D_m \) are solutions of the following coupled system of difference equations:
\[ D_{2m} = 1 + \alpha D_{2m+1} + (1 - \alpha)D_{2m-1} \]
\[ D_{2m+1} = 1 + (1 - \alpha)(D_{2m+2} + \alpha D_{2m}), \]
subject to boundary conditions which depend on the location and the parity of the boundaries \( L \) and \( U \).

Let us put \( X_m = D_{2m} \) and \( Y_m = D_{2m+1} \). Substituting the first equation into the second \( \textit{et vice versa} \) we obtain
\[ X_{m+1} - 2X_m + X_{m-1} = -\frac{2}{\alpha(1 - \alpha)} \]
\[ Y_{m+1} - 2Y_m + Y_{m-1} = -\frac{2}{\alpha(1 - \alpha)}. \]
These equations have the general solution
\[ X_m = D_{2m} = c_0 + c_1 m - \frac{m^2}{\alpha(1 - \alpha)} \]
\[ Y_m = D_{2m+1} = d_0 + d_1 m - \frac{m^2}{\alpha(1 - \alpha)} \]
where the constants \( c_0, c_1, d_0 \) and \( d_1 \) have to be determined from the boundary conditions. For brevity we confine ourselves to the case \( U = 2a, L = -2b \).

In this case we have \( X_{-b} = X_a = 0 \) and
\[ Y_{-b} = 1 + (1 - \alpha)X_{-b+1}, \quad Y_{a-1} = 1 + \alpha X_{a-1}. \]
This yields
\[ D_{2m} = \frac{1}{\alpha(1 - \alpha)} \left[ ab + m(a - b) - m^2 \right] \]
\[ D_{2m+1} = \frac{1}{\alpha(1 - \alpha)} \left[ ab + (a - b)(1 - \alpha) - (1 - \alpha)^2 + m(a - b - 2(1 - \alpha)) - m^2 \right]. \]
which implies
\[ D_0 = \frac{ab}{\alpha(1 - \alpha)}. \]
The other cases can be dealt with in a similar fashion.

6. **The busy period of the chemical queue**

Let us finally return to the problem which initiated this study, the busy period of a chemical queue, as it has been introduced in section 1. Let \( k_n(t; m) \) denote the joint probability and the probability density of the time interval between time zero when the system is in state \( m \geq 1 \) and the time \( t > 0 \), when the system becomes empty for the first time, \( n \) jobs have been served in this interval. Let an arrival occur with rate \( \lambda \) when the system is in an even state and with rate \( \mu \) when the system’s state is odd. Similarly, let departures occur with rate \( \mu \) if the system state is even and rate \( \lambda \) when the state is odd.

The embedded random walk is two-periodic and has jump probabilities
\[ \alpha = \frac{\lambda}{\lambda + \mu}. \]
Assume first that \( m = 2a \). If there are to be \( n \) services, then the number of arrivals must be \( n - 2a \) and the total number of jumps (= steps of the embedded walk) must be equal to \( 2n - 2a \). From these steps the last one is fixed, it is necessarily a departure, a down-step starting at altitude 1 and therefore it has
probability $\alpha$. The other $2n - 2a - 1$ steps occur according to a Poisson process with rate $\lambda + \mu$. The joint density $k_n(t; 2a)$ follows immediately from (21):

$$k_n(t; 2a) = e^{-(\lambda + \mu)t} \frac{\lambda^{2n-2a-1} \mu^{2n-2a-1}}{(2n - 2a - 1)!} \times n \geq 2a$$

$$\times \sum_{\ell \geq 0} \left( \frac{\mu}{\lambda} \right)^{2\ell + a} \left( \frac{n - a - 1}{\ell} \right) \left( \frac{n - a - 1}{\ell + a - 1} - \left( \frac{\mu}{\lambda} \right)^2 \left( \frac{n - a - 1}{\ell + a + 1} \right) \right).$$

If $m = 2a + 1$, we obtain by the same reasoning, using (22) and interchanging $\alpha \leftrightarrow 1 - \alpha$:

$$k_n(t; 2a + 1) = e^{-(\lambda + \mu)t} \frac{\lambda^{2n-2a-1} \mu^{2n-2a-2}}{(2n - 2a - 2)!} \times n \geq 2a + 1$$

$$\times \sum_{\ell \geq 0} \left( \frac{\mu}{\lambda} \right)^{2\ell + a} \left( \frac{n - a - 1}{\ell} \right) \left( \frac{n - a - 2}{\ell + a - 1} - \left( \frac{\mu}{\lambda} \right)^2 \left( \frac{n - a - 2}{\ell + a + 1} \right) \right).$$

For general $a > 0$ these formulas appear to be new. If we set $a = 0$ in the above formula, a further simplification is possible. Using (19) and again interchanging $\alpha \leftrightarrow 1 - \alpha$, we get

$$k_n(t, 1) = e^{-(\lambda + \mu)t} \frac{\lambda^{2n-3} \mu^2}{(2n - 2)!} \sum_{\ell \geq 0} \left( \frac{\mu}{\lambda} \right)^{2\ell} \left( \frac{n - 1}{\ell} \right) \left( \frac{n - 1}{\ell + 1} \right).$$

In this way we recovered the formula (24) given Connolly et al. (1997).

7. References


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