Partial pooling by independent firms with allocation according to contribution to pool

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\textbf{ABSTRACT}

We consider two firms which pool some of their inventory. The pool is created by the firms' contributions, and a firm's entitlement for an allocation from the pool (if needed) is a function of its contribution. Transshipment from the pool is costly, but the firms can benefit from reduced risk through inventory sharing using the pool. We analyze the resulting non-cooperative game. We prove existence of a Nash equilibrium and compare it to a model with centralized control. An appropriate compensation cost for using the other firms' contribution to the pool can induce the retailers to achieve centralized solutions. We also compare the optimal partial pooling strategy to the special cases of no pooling and complete pooling and discuss situations where it is likely that one of the special cases will be optimal. Numerical results confirm that in the prevalent practice of partial pooling the retailers can achieve higher expected profits than under no pooling or complete pooling and that there is a significant difference between a setting with independent players and a model of central control.

\section{Introduction}

When demands are uncertain, pooling inventories is a common practice in multi-location inventory management. Pooling allows to benefit from reduced effective demand risk, in particular reduced risks of facing stockouts and leftovers, which increases profits and service levels (Frieden, 1979). For example, instead of having local inventories in each demand location, inventories can be physically pooled at a central storage facility. It can be also achieved through virtual pooling (i.e. transshipments) by transshipping excess inventory to demand locations facing stockouts. Other prevalent forms of inventory pooling include strategies such as postponement, product substitution and component commonality.

Traditionally, work in that area assumed central control. But if the parties (say, retailers) involved are independent firms there is an issue of how much will they contribute to the pool, and how will pooled units be allocated if needed. That may be determined by bargaining. We shall, however, use a different approach: The parties that contribute to the pool, will be assumed to have precedence for the units they contributed and be able to use units contributed to the pool by the other party if not needed (Ben-Zvi and Gerchak, 2012; Gerchak, 2016). Note that Müller et al. (2002) proved that the core of this game is not empty; so all parties can benefit from pooling.

Here we consider the common practice scenario of physical centralization of inventories at a warehouse where, in addition to pooled units, the retailers can also keep units in their sites (i.e., at the retail level). That gives rise to partial pooling.

In a partial pooling system each retailer selects its own inventory and its contribution to the central warehouse (see Gerchak (2017) for a centrally controlled setting). Using a central warehouse for deliveries to retailers who are short, which is quite common, essentially implies that transshipment among retailers (virtual pooling) is not practical. Thus retailers who use a central warehouse in that way are examples of our scenario. Here it will be assumed that delivery of other firm's units at the warehouse involves payment and transshipments between retail stores is prohibitively expensive. Costs of initial placement of units in the pool and stores may be different, and the model also includes delivery costs from the pool to customers.

While the practice of partial pooling occurs quite often, it has been rarely studied in the literature. It is quite common that retailers hold some inventory at their sites for the part of demand that is (almost) certain to occur and additional pooled inventory at a central warehouse for the considerably uncertain part of demand. Stocked out retailers are then searching items at the warehouse when transshipments at the retail level are not possible or highly expensive. Such arrangement, for example, is given in the case of a shoe company described in Wee and Dada (2005), where the warehouse is the only source for replenishment in stockout situations. Narus and Anderson (1996) provide various

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practical examples for such flexible distribution channels. Kurata (2014) discusses the example of a parent company of Seven Eleven Japan that has established an online shopping system where customers can choose between home delivery or a pick-up at a conveniently located store. In such omni-channel systems an appropriate allocation of inventory between a central warehouse and the stores is a key issue (Kurata, 2014). Amrani and Khmelnitsky (2017) describe an example of maintenance of avionic systems where items are kept in a two-echelon supply chain. A large depot supplies a number of geographically dispersed bases facing stochastic demand that provide items from local inventory first. In a non-commercial setting partial pooling occurs for humanitarian organizations that usually have their own warehouses for relief items to respond to a disaster but can also store relief items at a depot of the UNHCR to benefit from risk pooling and save costs (Toyasaki et al., 2017).

Note that our work is the first to analyze a partial pooling system assuming independent retailers. As outlined in Wee and Dada (2005) partial pooling is especially appropriate in organizational structures such as franchising arrangements where independent retailers cooperate in terms of sharing a central warehouse. This actually supports our model assumption.

The aim of this paper is to analyze such a partial pooling model with two independent retailers, resulting in a non-cooperative game. Thereby, the key question is how retailers should optimally set their inventory levels at the stores and the central warehouse. First, the special cases of complete pooling and no pooling are discussed that allows us to benchmark the partial pooling system and to identify in which situation it is likely that one of two special cases will be optimal. We then provide a detailed description of a partial pooling inventory system and prove joint concavity of the expected profit functions under both centralized and decentralized control. Analytical results are complemented by numerical examples in order to study the performance of partial pooling for independent retailers.

The remainder of this paper is structured as follows. In Section 2 we briefly review relevant literature related to inventory pooling with independent players and to the concept of partial pooling. Section 3 describes the problem setting and the assumptions of our model. The special cases of no pooling and complete pooling which serve as benchmarks are analyzed in Section 4. The mathematical model of partial pooling and structural results complemented by numerical examples and managerial insights from our analysis are given in Section 5. Finally, Section 6 summarizes the findings and discusses potential future research directions.

2. Literature

Literature on risk pooling with independent retailers, constituting a non-cooperative game, mostly focuses on inventory sharing systems with lateral transshipments, i.e. inventory sharing between locations on the same echelon (e.g. Rudi et al., 2001; Hu et al., 2007; Shao et al., 2011; Arikian and Silbermayr, 2018) or substitution (e.g. Parlar, 1988; Lippman and McCardle, 1997; Netessine and Rudi, 2003; Cachon and Netessine, 2006). These risk pooling systems are different from our setting. Our focus is on physically centralizing inventory of the retailers into a single location. Further, we do not consider any demand substitution.

Literature focusing on physical pooling of inventories at a central warehouse mostly assume that inventories for multiple locations are managed centrally (e.g. Eppen, 1979; Chen and Lin, 1989; Gerchak and He, 2003; Corbett and Rajaram, 2006; Silbermayr et al., 2017) or assumes a cooperative newsvendor game (e.g. Hartman and Dror, 2005; Slikker et al., 2005; Fiestras-Janeiro et al., 2011). This may be due to the fact that for a system with more than two locations one needs a proper allocation rule for assigning the joint stock to the retailers, if several retailers are facing a stockout situation. Finding an appropriate allocation rule is a key issue in decentralized inventory systems with more than two locations. However, this is not the focus of this work.

Ben-Zvi and Gerchak (2012) consider physical pooling of inventories with two independent firms differing in their shortage cost. They assume that a retailer facing a stockout situation will receive an allocation proportional to its contribution to the pool. Gerchak (2016) analyze consequences of such a non-cooperative game with a modified scheme that is beneficial to all parties relative to a no pooling situation. Both papers assume that it is costless for retailers to use residual stock from the other retailers’ contribution to the joint stock. We, on the other hand, have the user of other’s contribution to the pool pay for that. Here we shall allow partial pooling, so each retailer has to choose its own inventory and its contribution to the pool.

Partial pooling has only been studied for the case of centrally managed inventory systems, except for Anupindi et al. (2001) who, in contrast to our approach, consider a model where independent retailers after demand is realized cooperatively determine how to share residual inventory and divide the profit from pooling according to an allocation rule specified at the beginning of the game. Wee and Dada (2005) consider a centrally managed partial pooling model with transshipments between the retailers and delivery from a warehouse. Hence, their model is a combination of physical pooling of inventories and lateral transshipments. They provide results pertaining to when it is optimal to open a warehouse. Amrani and Khmelnitsky (2017) study a partial pooling model similar to Wee and Dada (2005) where the total amount of inventory is fixed. Gerchak (2017) considers partial pooling as a combination of i) physical pooling and ii) decentralized storage at the retail locations, i.e. transshipments between retail stores is prohibitively expensive. Conditions are derived for a partial pooling solution to be locally superior to a no pooling solution. Kurata (2014) studies a model with one supplier and multiple identical retailers where items can be stocked at the retail level and at the supplier in order to find the optimal allocation of items in such a two-echelon inventory system.

In this paper we study a system of partial pooling as in Gerchak (2017), but we assume that the retailers are independent firms. Such a partial pooling system results in a non-cooperative game in which the retailers independently determine their inventory quantities at the store and the pool and have precedence for the units they contributed to the pool. Further, we will compare it to a centrally managed model and also discuss special cases of partial pooling (no pooling and complete pooling).

3. Problem description and assumptions

Consider two independent retailers, and , with continuous random demands , (for an example with discrete demand we refer to Appendix A).

The retailers have to choose their own stocks, , and , and their contributions to the common stock at the central warehouse, , and , that then becomes . We assume that the warehouse is the only source that delivers inventory to the retailers in case of a stockout, as transshipments between the retailers is prohibitively expensive (Gerchak, 2017). The cost of placing a unit at a retailer are , while the cost of placing a unit at the pool are . Hence, any difference in and can be due to asymmetries in ordering and storage cost between the central warehouse and a store at the retail level. For example, the cost of the common warehouse could be indirectly captured by the purchase cost at the pool being higher than the purchase cost at a retailer . Note that captures the initial investment for establishing the common facility. Any delivery from the pool incurs a transportation cost of per unit. So establishing a central warehouse will be only beneficial if the advantage of inventory pooling exceed these extra costs. The revenue per unit sold is . Thus both retailers are assumed to have same revenue and placing cost per unit.

The sequence of events is as follows. First, retailers place orders for their own inventories and their contributions to the pool.
simultaneously. Then, demands realize and each retailer uses its own stock from the retail location and the pool. If demand of retailer $i$ exceeds its own total stock $q_i + Q$, then it may use retailer $j$’s surplus stock at the pool at the per unit compensation cost $r$. The same holds for retailer $j$.

That is, we assume that each retailer has precedence over its contribution to the pool. If it does not need (part of) it, the other retailer can use it. If retailer $i$ uses a portion of retailer $j$’s surplus pool $Q$, it has to pay a compensation cost of $r$ per unit (the same holds for retailer $j$ if it uses a portion of $Q$).

To avoid trivial solutions we assume that $r > r + t$ and $r \in [0, r - t]$. The notation is summarized in Table 1 and an overview of the model is illustrated in Fig. 1.

We presume that the compensation cost $r$ is defined in advance which can be also observed in practice for such cooperative arrangement (Narus and Anderson, 1996). The compensation cost for using the other retailers surplus stock at the pool will have a direct implication on retailers’ profitability, consequently a discussion on how this compensation cost $r$ is set is needed. It may be either set by some external authority (such as e.g. a common supplier or third party logistic provider) or it needs to be determined at the outset, plausibly by bargaining (for example, Nash bargaining). In the later case, for retailer $i$ and $j$ with disagreement value, $d_i$ and $d_j$, which will be its expected no pooling profit one will be using the value of $r$ which will maximize the product $[Eu_i(r) - d_i][Eu_j(r) - d_j]$, with $Eu_i(.)$ and $Eu_j(.)$ being the expected utility functions representing the preferences of retailer $i$ and $j$.

That cooperative phase and the subsequent non-cooperative game will constitute a “bi-form” game (see Hanany et al. (2006)).

Hence, in the following we examine the non-cooperative inventory decisions of partial pooling for any $r \in [0, r - t]$, that is either exogenously given or alternatively already set by bargaining at the foregoing cooperative phase of the game (similar to the approach of e.g. Rudi et al. (2001) and Shao et al. (2011) where, however, only local storage with pooling via transshipments is considered).

The compensation cost plays an important role in determining the benefits of (partial) pooling for each retailer (see Sections 4.2 and 5.3 where we discuss the impact of varying $r$). Hence, we will compare our model to a centrally controlled model where a single decision maker is maximizing the expected aggregated profit and consequently compensation costs do not matter.

As for the inventory policies of the retailers we distinguish between the following cases:

- $Q = 0 \Rightarrow$ No Pooling (NP) with policy $(q_i^{NP}, q_j^{NP}, 0)$
- $q_i = q_j = 0 \Rightarrow$ Complete Pooling (CP) with policy $(0, 0, Q^{CP})$
- $q_i, q_j > 0 \Rightarrow$ Partial Pooling with policy $(q_i, q_j, Q)$

In case of a centralized decision maker, it is well known that, without transportation costs $t$ and equal purchase costs, the expected profits satisfy $\Pi(0, 0, Q^{CP}) \geq \Pi(q_i^{NP}, q_j^{NP}, 0)$ always (e.g., Eppen, 1979; Gerchak and He, 2003).

Recently it was shown by Gerchak (2017), in a centralized scenario with transportation costs from the pool, that it is possible that $\Pi(q_s^*, q_j^*, Q) > \Pi(0, 0, Q^{CP})$ and $\Pi(q_i^*, q_j^*, Q^*) > \Pi(q_i^{NP}, q_j^{NP}, 0)$ for $q_i^*, q_j^*, Q^* > 0$, so partial pooling can be strictly optimal. Note that if one wishes the partial pooling solution to be better than no pooling for every demand realization one needs to have $q_i + q_j > Q \geq q_i^{NP} + q_j^{NP}$ (see Hartman and Dror, 2005). We shall not impose that requirement.

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**Table 1**

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_i$</td>
<td>stochastic demand of retailer $i$ with distribution function $F_i$</td>
</tr>
<tr>
<td>$(q_i^{NP}, q_j^{NP})$</td>
<td>(optimal) inventory quantity of $i$ placed at central location/pool</td>
</tr>
<tr>
<td>$(q_i^{CP}, q_j^{CP})$</td>
<td>(optimal) inventory quantity of $i$ at retail location $i$</td>
</tr>
<tr>
<td>$q_i^{CP}$</td>
<td>optimal inventory quantity of $i$ placed at central location/pool under special case of complete pooling</td>
</tr>
<tr>
<td>$q_i^{NP}$</td>
<td>optimal inventory quantity of $i$ at retail location $i$ under special case of no pooling</td>
</tr>
<tr>
<td>$r$</td>
<td>per unit revenue</td>
</tr>
<tr>
<td>$c_p$</td>
<td>per unit purchase cost at the pool</td>
</tr>
<tr>
<td>$c_u$</td>
<td>per unit purchase cost at the retail locations</td>
</tr>
<tr>
<td>$t$</td>
<td>per unit transportation cost from the pool to retailers</td>
</tr>
<tr>
<td>$\tau$</td>
<td>per unit compensation cost for using the competitor’s pool</td>
</tr>
<tr>
<td>$n_i$</td>
<td>expected profit of retailer $i$</td>
</tr>
<tr>
<td>$n_i^*$</td>
<td>expected profit of retailer $i$ under special case $k \in {CP, NP}$</td>
</tr>
<tr>
<td>$\Pi^t$</td>
<td>total expected profit of retailer $i$ and $j$</td>
</tr>
</tbody>
</table>

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Fig. 1. Partial pooling system.
4. No pooling and complete pooling

First, we will consider the two extreme special cases of partial pooling - no pooling and complete pooling - which are easier to analyze, and provide us a benchmark for evaluating any partial pooling.

4.1. No pooling

If \( Q = 0 \) then there is no inventory pooling involved in the system (see Fig. 1) and the problem for retailer \( i \) reduces to the standard newsvendor problem. No expenditures associated with a warehouse are incurred. The expected profit of retailer \( i \) is

\[
\pi_i^{NP} = (r - c_p)q_i - tE(q_i - X_i)^+, \tag{1}
\]

where \( X_i^+ = \max(X_i, 0) \).

The expected profit maximizing inventory \( q_i^{NP} \), \( i = 1, 2 \) satisfies

\[
F_X(q_i) = \frac{r - c_p}{r}, \tag{2}
\]

where \( \frac{r - c_p}{r} \) is the critical ratio of the standard newsvendor problem.

4.2. Complete pooling

If \( q_i = q_j = 0 \), then the two independent retailers have to only choose their contributions to the common stock, \( Q_i \) and \( Q_j \), that then becomes \( Q = Q_i + Q_j \). Here \( c_p \) is irrelevant and stores can eliminate their storage space. In this case each sale will require a delivery from the pool to the retail locations at cost \( t \). Supplying retailer \( i \) from \( j \)'s pool additionally incur a per unit compensation cost \( r \). We refer to this transfer as \( \text{cross}_j \). Note that even if \( r = 0 \) a retailer has an incentive to contribute to the pool (rather than 'only' on the other retailer's contribution), as then larger demands can be satisfied, increasing the expected revenue.

To obtain the expected profit of retailer \( i \) under continuous demand we consider eight distinct cases.

1. If \( x_i \leq Q_i \) and \( x_j \leq Q_j \), \( \text{sales}_i = x_i \), \( \text{cross}_j = \text{cross}_j = 0 \)
2. If \( Q_i \leq x_i \leq Q_i + Q_j \) and \( Q_i + Q_j > x_i + x_j \), \( \text{sales}_i = x_i \), \( \text{cross}_j = x_j - Q_j \), \( \text{cross}_j = 0 \)
3. If \( Q_i \leq x_i \leq Q_i + Q_j \), \( Q_i + Q_j \leq x_i + x_j \) and \( Q_j > x_j \), \( \text{sales}_j = Q_j - x_j \), \( \text{cross}_j = Q_i - x_i \), \( \text{cross}_j = 0 \)
4. If \( x_i > Q_i + Q_j \) and \( Q_i > x_j \), \( \text{sales}_i = Q_i + Q_j - x_j \), \( \text{cross}_j = Q_j - x_j \), \( \text{cross}_j = 0 \)
5. If \( x_i \geq Q_i \) and \( x_j \geq Q_j \), \( \text{sales}_i = Q_j \), \( \text{cross}_j = \text{cross}_j = 0 \)
6. If \( x_i \leq Q_i \) and \( x_j > Q_j \) and \( Q_i + Q_j > x_i + x_j \), \( \text{sales}_i = x_i \), \( \text{cross}_j = x_j - Q_j \), \( \text{cross}_j = 0 \)
7. If \( x_i \leq Q_i \), \( Q_i + Q_j \leq x_i + x_j \) and \( x_j \leq Q_j \), \( \text{sales}_i = x_i \), \( \text{cross}_j = Q_i - x_i \), \( \text{cross}_j = 0 \)
8. If \( x_i \leq Q_i \), \( Q_i + Q_j \leq x_i + x_j \) and \( x_j > Q_j \), \( \text{sales}_i = x_i \), \( \text{cross}_j = 0 \), \( \text{cross}_j = Q_i - x_i \)

Then expected sales and cross-delivery are

\[
E(\text{sales}_i) = E(\min(X_i, Q_i)) + E(\text{cross}_j),
\]

\[
E(\text{cross}_j) = E(\min(X_i - Q_i)^+, (Q_j - X_j)^+) \),
\]

\[
E(\text{cross}_j) = E(\min((X_i - Q_i)^+, (Q_j - X_j)^+) \}
\]

The expected profit function of \( i \) is

\[
\pi_i^{CP} = -c_pQ_i + E[(r - t) (Q_i - (Q_i - X_i)^+) + (r - t)\min((X_i - Q_i)^+, (Q_j - X_j)^+) + r\min((X_i - Q_i)^+, (Q_j - X_j)^+) \}
\]

which is similar to the expected profit of a two location transshipment systems with decentralized storage studied e.g. in Rudi et al. (2001). The expected profit of \( i \) is concave in \( Q_i \) and the first order condition of \( i \) is:

\[
r - t - c_p - (r - t)F_{X_i < (Q_i - X_j)^+}(Q_i) - tf_{X_i > (Q_i - X_j)^+}(Q_i) = 0. \tag{3}
\]

There exists a unique Nash equilibrium \((Q_i^{CP}, Q_j^{CP})\) which is characterized by (3) (Rudi et al., 2001, Proposition 1).

Lemma 1. The retailers’ optimal inventory levels \(Q_i^{CP}\) and \(Q_j^{CP}\) are increasing in the compensation cost \(r\).

That is, a decentralized retailer \( i \) increases its stock with increasing compensation cost \( r \) paid to \( j \) for using his pool, as retailer \( i \)'s stockout cost increase while his leftover cost decrease. This is illustrated in Fig. 2 for the case of two independent gamma distributed demands with mean \( 10 \) and a coefficient of variation of \( 0.5 \). Note that the results throughout the paper are obtained via simulation optimization.

If the system is controlled by a central decision maker then complete pooling reduces to a problem with a single decision variable. The total expected profit from ordering \( Q \) units to the pool is given by

\[
\Pi^{CP,c} = (r - t - c_p)Q - (r - t)E(Q - X_i - X_j)^+ \}
\]

The expected profit is concave and the optimal order quantity under centralized control \(Q_i^{CP,c}\) is found by solving the first order condition

\[
r - t - c_p - (r - t)F_{X_i < (Q_i - X_j)^+}(Q_i) = 0. \tag{4}
\]

Hence, \(Q_i^{CP,c} = F_{X_i < (Q_i - X_j)^+}(\frac{r - t - c_p}{r - t - c_p})\) where \(\frac{r - t - c_p}{r - t - c_p}\) is the critical ratio of the (centralized) complete pooling scenario.

Proposition 1. For \( r = 0 \) the total inventory in the decentralized system is lower than in the centralized systems, i.e. \(Q_i^{CP} + Q_j^{CP} < Q_i^{CP,c}\).

An immediate consequence of Proposition 1 is that the decentralized system is not coordinated if \( r = 0 \), i.e. the retailers do not make system optimal (i.e. centralized) decisions. The retailers' inventory levels are too low. Since the retailers' optimal inventory levels are increasing in \( r \) (Lemma 1) we propose \( r \) as a mechanism to coordinate the decentralized system. In other words, we want to find a particular compensation cost \( r \) that induces the retailers to make system optimal inventory decisions. The result is given in the following proposition.

Proposition 2. The coordinating compensation cost that results in the highest expected profit under complete pooling is given by

\[
r^* = \left( \frac{r - t)F_{X_i < (Q_i - X_j)^+}(Q_i) - (r - t)F_{X_i > (Q_i - X_j)^+}}{F_{X_i < (Q_i - X_j)^+}(Q_i) - F_{X_i > (Q_i - X_j)^+}} \right. + \frac{Q_j^{CP,c} + Q_i^{CP,c}}{Q_i^{CP,c} + Q_j^{CP,c}}. \tag{5}
\]

For symmetric retailers with \(F_{X_i} = F_{X_j}\) we know that
Q^{\text{CP},e} = Q^{\text{CP},c} = Q^{\text{CP},c/2}. Hence, in the case of two symmetric retailers the coordinating compensation cost can be uniquely determined and always exists (Hu et al., 2007, Theorem 1). The coordinating compensation cost with varying transportation cost and order cost $c_F$ are reported in Table 2 (again for the case of two independent gamma distributed demands with mean 10 and a coefficient of variation of 0.5). We see that,

**Observation 1.** The coordinating compensation cost $r^*$ given in (5) is increasing in $c_F$ and decreasing in $t$.

### 5. Partial pooling

In this section we analytically and numerically discuss the partial pooling model discussed in Section 3.

#### 5.1. Partial pooling: decentralized control

For partial pooling with two independent retailers we consider ten distinct cases listed below in order to obtain the expected profit of retailer $i$. A graphical illustration of the possible cases is given in Fig. 3. We define $T_i$ as the number of units retailer $i$ uses from its own contribution to the pool. Then:

1. If $x_i \leq q_i$ and $x_j \leq q_j + Q_i$, $\text{sales}_i = x_i$, $\text{cross}_i = \text{cross}_j = 0$, $T_i = 0$
2. If $q_i \leq x_i \leq q_i + Q_i$ and $x_j \leq q_j + Q_j$, $\text{sales}_i = x_i$, $\text{cross}_j = \text{cross}_j = 0$, $T_i = x_j - q_j$
3. If $q_i \leq x_i \leq q_i + Q_i$ and $x_j \leq q_j + Q_j$, $\text{sales}_i = x_i$, $\text{cross}_j = x_j - q_j$, $T_i = 0$, $T_j = x_i - q_i$
4. If $x_i \leq q_i$ and $q_j + Q_j \leq x_j \leq q_j + Q_j + Q_i$, $\text{sales}_i = x_i$, $\text{cross}_j = x_j - q_j - Q_j$, $T_i = 0$, $T_j = x_i - q_i$
5. If $x_i \leq q_i$ and $q_j + Q_j \leq x_j \leq q_j + Q_j + Q_i$, $\text{sales}_i = x_i$, $\text{cross}_j = Q_i$, $T_i = 0$, $T_j = 0$
6. If $x_i \leq x_j \leq q_j + Q_i$ and $x_j \leq q_j + Q_j + Q_i$, $\text{sales}_i = x_i$, $\text{cross}_j = q_j + Q_j - x_i$, $T_i = 0$, $T_j = x_i - q_i$
7. If $x_i \leq x_j \leq q_j + Q_j + Q_i$ and $x_j \leq q_j + Q_j + Q_i - x_i$, $\text{sales}_i = x_i$, $\text{cross}_j = 0$, $T_i = x_j - q_j$, $T_j = Q_i$
8. If $q_j + Q_j \leq x_i$ and $x_j \leq q_j + Q_j + Q_i$, $\text{sales}_i = q_j + Q_j$, $\text{cross}_j = x_i - q_j - Q_j$, $T_i = 0$, $T_j = Q_i$
9. If $q_j + Q_j \leq x_i$ and $x_j \leq q_j + Q_j + Q_i$, $\text{sales}_i = q_j + Q_j + Q_i - x_i$, $x_j \leq q_j + Q_j + Q_i - x_i$, $\text{cross}_i = 0$, $\text{cross}_j = Q_i$, $T_i = Q_j$

### Table 2

<table>
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<tr>
<th>$Q$</th>
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<th>0.3</th>
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<td>0.5</td>
<td>0.81</td>
<td>0.79</td>
<td>0.77</td>
<td>0.75</td>
</tr>
</tbody>
</table>

#### 5.2. Partial pooling: central control

If the partial pooling system is centrally managed the decisions of such a system are $q_i$, $q_j$ and $Q$. The total expected delivery from the pool to the retailers is given by

$$E(T) = E(\min(Q, (X_i - q_i)^+ + (X_j - q_j)^+)).$$

The total expected profit under centralized control is

$$\Gamma^* = -c_F(q_i + q_j) - c_FQ + rE(\min(Q, (X_i + q_j) + E(\min(Q, (X_j + q_j)))) + (r - t)E(\min(Q, (X_i - q_i)^+ + (X_j - q_j)^+)).$$

From this we have the following.

**Proposition 4.** 1) For the centralized problem the first order necessary conditions are

![Fig. 3. Ten distinct cases of partial pooling.](image-url)
\[ \begin{align*}
\frac{\partial \pi_i}{\partial q_i} &= r - c_B - (f_X(q_i) - (r - t) Pr((X_i - q_i)^x + (X_j - q_j)^x < Q, q_j < X_j) = 0 \quad (10) \\
\frac{\partial \pi_i}{\partial Q} &= r - t - c_F - (r - t) Pr((X_i - q_i)^x + (X_j - q_j)^x < Q) = 0. \quad (11)
\end{align*} \]

ii) For symmetric retailers the total expected profit under centralized control given in (9) is jointly concave in \((q_i, q_j, Q)\).

Hence, for symmetric retailers there is a global optimal solution \((q_i^{\ast}, q_j^{\ast}, Q^{\ast})\) maximizing (9), found by solving the system of first order conditions (10) and (11).

5.3. Numerical analysis

We perform a numerical analysis to gain additional insights into the partial pooling model. In particular, we are interested how the compensation cost which has been specified in advance of the inventory game, affects the optimal order quantities and profits. We want to compare the model with individual decision making to a centrally controlled model and discuss how demand correlation affects the retailers partial pooling quantities.

We assume that the demands are gamma distributed with a mean of 10 and a coefficient of variation of 0.5. Unless otherwise stated, we assume independent demands and retail price and purchase cost at the pool are fixed to \(r = 2\) and \(c_F = 0.3\).

5.3.1. Impact of cost parameters on optimal order quantities and expected profits

An overview of the optimal partial pooling strategy varying purchase cost \(c_B\), transportation cost \(t\) and compensation cost \(\tau\) is given in Table 3 where (simulated) optimal order quantities are rounded. From Table 3 we can make the following intuitive observations.

Observation 2. For the models of decentralized and centralized control, with decreasing \(t\), the optimal inventory level at the pool increases while the optimal inventory level at the retail locations decreases.

Our numerical results show that the analytical results of the complete pooling case also hold for partial pooling which is also intuitive:

Observation 3. The retailers increase their contribution to the pool with increasing compensation cost \(\tau\).

Again \(r\) can be used as a mechanism to coordinate the partial pooling system, i.e. inducing the retailers to make centralized inventory decisions.

If we compare the optimal policy of decentralized control with that of centralized control, we see that generally when \(r\) is not that high the total inventory quantity of the pool is higher under centralized control, i.e. \(Q^* + Q^j < Q^c\) where \(Q^j = Q^j(\tau)\) (see Table 3). This result is intuitive, as the decentralized retailer cannot benefit from full risk sharing which would be possible under centralized control. Consequently, the retailers are more cautious and rely more on their local inventories at the store.

Figs. 4 and 5 show the impact of increasing \(r\) on the optimal expected profits of \(i\) and \(j\), respectively, exemplary for two scenarios: \(r = 2, c_F = 0.3, c_B = 0.3, \tau = 0.1\) and \(r = 2, c_F = 0.3, c_B = 0.4, \tau = 0.2\). Comparing the expected profits of the decentralized system with the centralized system the numerical results confirm the following (see Fig. 5 b):

Observation 4. An appropriate compensation cost \(\tau\) can coordinate the partial pooling system.

Further, comparing partial pooling with no pooling or complete
Observation 5. The retailers can achieve higher expected profits under partial pooling than under no pooling or complete pooling. Consequently partial pooling can be indeed optimal in a decentralized system and the benefits from partial pooling can be significant (see Figs. 4 and 5). This result illustrate the practical relevance of such a partial pooling system which probably occurs quite often in practice.

5.3.2. Property of optimal solution: boundary vs. inner solution

For relatively low transportation cost $t$ and compensation cost $r$, respectively, it is likely that the optimal policy in the decentralized model is a boundary solution, i.e. either no pooling is optimal if $c_R$ is low compared to $c_P$ or complete pooling is optimal if $c_R$ is high compared to $c_P$ (see Table 3). In the following we analyze, how likely it is to observe a boundary solution. We consider an extended numerical design shown in Table 4 where we analyze high-profit product settings as well as low-profit product settings (i.e. newsvendor ratios $\frac{c_P}{c_R}$ more and less than $F(\mu)$, respectively).

<table>
<thead>
<tr>
<th>Table 4</th>
<th>Summary of parameter values with in total 150 instances.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>2</td>
</tr>
<tr>
<td>$t$</td>
<td>(0.1,0.2,0.3)</td>
</tr>
<tr>
<td>$r$</td>
<td>(0.0,0.3,0.6,0.9,1.2)</td>
</tr>
<tr>
<td>$c_P$</td>
<td>high-profit setting</td>
</tr>
<tr>
<td>$c_R$</td>
<td>low-profit setting</td>
</tr>
<tr>
<td>$\tau$</td>
<td>0.3</td>
</tr>
<tr>
<td>$\tau$</td>
<td>1.5</td>
</tr>
<tr>
<td>$c_P$</td>
<td>(0.2,0.25,0.3,0.35,0.4)</td>
</tr>
<tr>
<td>$c_R$</td>
<td>(1.4,1.45,1.5,1.55,1.6)</td>
</tr>
<tr>
<td>$\tau$</td>
<td>[0.8,0.9]</td>
</tr>
<tr>
<td>$\tau$</td>
<td>[0.2,0.3]</td>
</tr>
</tbody>
</table>

Table 5 | Number of numerical instances (in %) where NP, CP or PP is optimal. |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>NP optimal</td>
<td>CP optimal</td>
</tr>
<tr>
<td>high-profit setting</td>
<td>decentralized model</td>
</tr>
<tr>
<td>central model</td>
<td>0%</td>
</tr>
<tr>
<td>low-profit setting</td>
<td>decentralized model</td>
</tr>
<tr>
<td>central model</td>
<td>0%</td>
</tr>
</tbody>
</table>

For the models of decentralized and centralized control, Table 5 summarizes whether a boundary solution (NP or CP) or an inner solution is optimal. Interestingly we see from Table 5 that:

**Observation 6. The solution is more often inner one with a centralized control than with a decentralized one.**

Indeed, the difference between $c_R$ and $c_P$ has to be very large in order to obtain situations where a boundary solution is optimal in a model with central control. Further, comparing high-profit and low-profit settings from Table 5 we observe that for the decentralized model:

**Observation 7. In a high-profit product setting it is more likely that complete pooling is optimal compared to a low-profit product setting.**

To summarize, we see that there is a significant difference between the decentralized and centralized model. Hence, local decision making does affect the optimal inventory levels at the retailers and the pool, for example, it induces the retailers to rely more on their inventories at the retail level.

5.3.3. Impact of demand correlation

So far our numerical examples are based on the assumption that demands are independent. As demand dependence is a critical issue in inventory pooling we illustrate the impact of correlation on the optimal order quantities in Fig. 6. We use a Gaussian copula for the joint distribution with gamma distributed marginals (mean 10 and coefficient of variation 0.5 as in the examples before). This implies a linear dependence (i.e. correlation) between the demands that is expressed by $\rho$. For more details on copulas we refer to e.g. Silbermayr et al. (2017). We observe the following:

**Observation 8. For the model of decentralized and centralized control, the inventory level at the retailer $q_i^* \text{ increases}, while the inventory level at the pool $Q^*$ decreases with increasing correlation coefficient $\rho$.**

The benefit of pooling decreases as $\rho$ increases, consequently the retailers will increase the order quantities at the retail locations. As $\rho$ approaches 1, the inventory at the pool approaches zero, while the inventory at the retailers approach $q_i^{NP}$. However, from Fig. 6, we see that the decreasing effect of $\rho$ on the inventories is weaker in the decentralized system than in the centralized system. Additionally, while for $\rho = 0.9$ in Fig. 6 the decentralized retailers will contribute nothing to the pool (i.e. no pooling is optimal), in the centralized system it is still optimal to store a part of the inventory in the pool. This implies that for independent retailers it will be more likely that the optimal
partial pooling policy will result in no pooling, especially when demand correlation is very high.

5.4. Managerial insights

From the numerical examples the following managerial insights based on the modelling framework of partial risk pooling introduced in Section 3 can be derived.

From the analysis in Section 5.3.1, it can be observed that the practice of partial pooling can be more beneficial for the retailers than no pooling or complete pooling (i.e. no local storage). Hence, the collaborative activity of physically pooling part of the inventory in order to deal with high demand uncertainty can be beneficial for independent retailers. The retailers should increase their inventory levels at the pool (and at the same time decrease their inventory at the retail levels) as the unit order cost at the retail level increases or the unit transportation cost from the warehouse to the customers decreases (see Table 3 and Observations 2 and 5).

From the managerial perspective, it is important to notice that the retailers should increase their inventory quantities at the warehouse as the per unit compensation cost for using the other retailers surplus stock increases (see Table 3 and Observation 3). Indeed, if the compensation cost is set appropriately then the partial pooling system can induce decentralized retailers to achieve the centralized optimal solution where the aggregated profit is maximized (Fig. 5 and Observation 4). Consequently, in practice given the important role of the compensation cost in the system, decision makers that are able to control this cost should be aware that the compensation costs, appropriately set, can be used as a mechanism to coordinate such a decentralized system.

In Section 5.3.2 we quantitatively show that there can be a significant difference between the situation of individual and central decision making (Observation 6). The reason is that, independent decision making implies that the retailers are not able to fully exploit the risk pooling benefit. As a result the retailers will rely more on their inventory at the local storage facilities.

The analysis in Section 5.3.3, where we discuss the impact of demand correlation between independent retailers shows the well known result that the advantage of risk pooling decreases as the correlation between the demands increases. The optimal inventory at the warehouse decreases as correlation increases (Observation 8) and approaches zero as correlation approaches one. However, independent retailers maximizing their own profits will reach a no pooling strategy faster compared to a system managed by a central decision maker maximizing the aggregate profit (see Fig. 6). This is again due to the fact that independent decision maker maximizing their own profits are not able to fully exploit the pooling benefit. It is important to recognize such inefficiencies in practice situations in order to being able to improve performance of the whole supply chain.

To summarize, our analysis provides first insights into the benefits of a partial pooling system with inventory managers acting individually in order to understand the impact of the system’s cost parameters and demand correlation structure on the optimal inventory levels and associated profits compared to a centrally coordinated system.

6. Conclusion

Motivated by an inventory pooling strategy that is quite prevalent in practice, we have developed a formal model of partially pooling inventories, where two independent firms pool some of their inventory which can be shared at a central warehouse, while keeping the rest at their local storage facilities. Our model appears to be the first contribution to this field and as obtaining analytical results is difficult, we display numerical results. We prove existence of a Nash equilibrium in such an inventory system and analyze the performance of partial pooling by comparing it to a model with centralized control. We show that local decision making instead of central decision making strongly impacts the optimal inventory policy in a partial pooling scenario. However, we find that setting an appropriate compensation cost for using the other firms contribution to the pool can induce the retailers to achieve system optimal (centralized) solutions. We also compare the optimal partial pooling strategy to the special cases of no pooling and complete pooling and discuss situations where it is likely that one of the special cases will be optimal. Our results confirm that partial pooling can indeed significantly outperform no pooling and complete pooling, which indicates the practical relevance of pooling some, but not all, inventories at a central storage location.

In our model we have assumed that lateral transshipments between the retailers are not possible. However, the model could be easily extended to any situation where delivering an item from the warehouse to a retailer is less expensive than lateral transshipments, but lateral transshipments are still beneficial (i.e. less costly than the retailers revenue). For the sequence of events it would imply that transshipments occur only whenever the inventory at the warehouse has been fully used up. The structural results discussed in this paper, however, would still hold in such situations.

In this paper we have focused on a system with two independent firms. Studying a system with more than two independent players and finding appropriate allocation rules for sharing the pool is a promising research field.

Appendix A. Discrete demand example

Suppose that \( X_i \) and \( X_j \) are independent and identically distributed, with

\[
X_i = \begin{cases} 
0, & \text{w.p. } p \\
1, & \text{w.p. } s \\
2, & \text{w.p. } 1 - p - s,
\end{cases} \tag{12}
\]

\( i = 1, 2, \ p, s > 0 \) and \( p + s \leq 1 \). This example was considered by Gerchak (2017) for a centralized system.
Assume a symmetric equilibrium \((q, Q)\). For each \((q, Q)\) used by a retailer, we check conditions on the parameters \((r, c_R, c_p, p, s)\) such that the same \((q, Q)\) is optimal for the other retailer.

We provide only a summary of the conditions.

We have the following possible (symmetric) Nash equilibria \(N = (q, Q)\) and resulting expected profits \(E(\pi)\) of a retailer:

1. \(N_1 = (0,0)\) and \(E(\pi_1) = 0\)
2. \(N_2 = (1,0)\) and \(E(\pi_2) = r(1 - p) - c_R\)
3. \(N_3 = (0,1)\) and \(E(\pi_3) = r((1 - p - s)2p + (1 - p - s)(1 - p) + s) - c_p + sp(1 - p - s) - r(1 - p - s)p = r(1 - ps - p^2) - c_p\)
4. \(N_4 = (1,1)\) and \(E(\pi_4) = r((1 - p - s)2p + (1 - p - s)(1 - p) + s - c_q - r(2 - 2p - s) - c_R - c_p\)
5. \(N_5 = (2,0)\) and \(E(\pi_5) = r(2 - 2p - s) - 2c_R\)
6. \(N_6 = (0,2)\) and \(E(\pi_6) = r(2 - 2p - s) - 2c_p\)

Thus,

1. \(((0,0), (0,0))\) is an equilibrium if: \(c_R \geq c_p\) and \(r < \frac{c_p}{1 - p - s - p - s}\) or \(c_R \leq c_p\) and \(r < \frac{c_p}{1 - p}\)

2. \(((1,0), (1,0))\) is an equilibrium if \(c_R \leq c_p\) and \(r < \frac{c_p}{1 - p - s - p - s}\) or \(c_R \leq c_p\) and \(r < \frac{c_p}{1 - p}\)

3. \(((0,1), (0,1))\) is an (altruistic) equilibrium if: \(c_R \leq c_p\) and \(r < \frac{c_p}{1 - p - s - p - s}\) or \(c_R \leq c_p\) and \(r < \frac{c_p}{1 - p}\)

4. \(((1,1), (1,1))\) is not an equilibrium, as \((1,1)\) cannot be a better response than both \((2,0)\) and \((0,2)\) since one requires \(c_p < c_R\) and the other \(c_p > c_R\)

5. \(((2,0), (2,0))\) is an equilibrium if: \(c_R \leq c_p\) and \(r > \frac{c_p}{1 - p - s - p - s}\)

6. \(((0,2), (0,2))\) is an equilibrium if: \(c_R \leq c_p\) and \(r > \frac{c_p}{1 - p - s - p - s}\)

We can summarize the results as follows:

#### Appendix B. Proofs

**Proof of Lemma 1.** This immediately follows from (3).

**Proof of Proposition 1.** We know that \(F_{X_R}(Q^F) = r \frac{e^{t - q - q}}{r^t}Q^2\). Now for \(t = 0\) we have: \(\pi_t = (t - t)E\min(X_t, Q_t) + (r - t)E\max(X_t) - c_PQ_t\). The expected quantity cross-delivered can be written as (see e.g. Dong and Rudi, 2004; Shao et al., 2011):

\[
E\max(X_t) = \text{E}(\min(X_t, Q_t) - \text{E}(\min(X_t, Q_t) - \text{E}(X_t)).
\]

Hence, 

\[
E(X_t) = E\max(X_t) = \text{E}(\max(X_t, Q_t) - \text{E}(\max(X_t, Q_t) - \text{E}(X_t)).
\]

We can rewrite the retailers expected profit as 

\[
\pi_t = (t - t)E\min(X_t, Q_t) + (r - t)E\max(X_t, Q_t) - (r - t)E\min(X_t, Q_t) - (r - t)E\max(X_t) - E\max(X_t) - E\max(X_t).
\]

Now, the first derivative is 

\[
\frac{\partial \pi_t}{\partial r} = (t - t)(1 - F_{X_t}(Q + Q)) - (r - t)(P(X_t < Q, X_t > Q) + Q) - c_R = 0.
\]

Consequently, 

\[
F_{X_t}(Q_t) = \frac{1 - \frac{e^{-t}}{r^t}}{1 - \frac{e^{-t}}{r^t}} = P(X_t < Q, X_t > Q) + Q < Q^F.
\]

**Proof of Proposition 2.** \(r^*\) is found by equating the left-hand side of (3) evaluated at \(Q^{CP,F} = Q^{CP,F} + Q^{CP,F}\) with the left-hand side of (4).

**Proof of Proposition 3.** The expected profit function can be written as 

\[
i_t = \int_{q=0}^{q=1} q^{q+1} h(q) r(\int_{x=0}^{x=q} q^{q+1}(x) \, dx) \, dq + \int_{q=0}^{q=1} q^{q+1} h(q) r(\int_{x=q}^{x=1} q^{q+1}(x) \, dx) \, dq
\]

\[
+ \int_{q=0}^{q=1} q^{q+1} h(q) r(\int_{x=q}^{x=1} q^{q+1}(x) \, dx) \, dq
\]

\[
+ \int_{q=0}^{q=1} q^{q+1} h(q) r(\int_{x=q}^{x=1} q^{q+1}(x) \, dx) \, dq
\]

\[
+ \int_{q=0}^{q=1} q^{q+1} h(q) r(\int_{x=q}^{x=1} q^{q+1}(x) \, dx) \, dq
\]

\[
+ \int_{q=0}^{q=1} q^{q+1} h(q) r(\int_{x=q}^{x=1} q^{q+1}(x) \, dx) \, dq
\]

\[
+ \int_{q=0}^{q=1} q^{q+1} h(q) r(\int_{x=q}^{x=1} q^{q+1}(x) \, dx) \, dq
\]

\[
+ \int_{q=0}^{q=1} q^{q+1} h(q) r(\int_{x=q}^{x=1} q^{q+1}(x) \, dx) \, dq
\]

\[
+ \int_{q=0}^{q=1} q^{q+1} h(q) r(\int_{x=q}^{x=1} q^{q+1}(x) \, dx) \, dq
\]

\[
+ \int_{q=0}^{q=1} q^{q+1} h(q) r(\int_{x=q}^{x=1} q^{q+1}(x) \, dx) \, dq
\]

In order to prove that \(i_t\) is jointly concave in \((q, Q)\), it is necessary to show that the determinants of \(H\) alternate in sign (starting from negative).

\[
H = \begin{bmatrix}
\frac{\partial^2 i_t}{\partial q^2} & \frac{\partial^2 i_t}{\partial q \partial Q} \\
\frac{\partial^2 i_t}{\partial Q \partial q} & \frac{\partial^2 i_t}{\partial Q^2}
\end{bmatrix}
\]

Now,
\[ \frac{\partial \rho}{\partial q} = \int_{x=0}^{q} \int_{x=0}^{q} f(x, y) dx dy + \int_{x=0}^{\infty} \int_{y=0}^{\infty} f(x, y) dx dy + \int_{x=0}^{\infty} \int_{y=0}^{\infty} f(x, y) dx dy \]

and

\[ \frac{\partial \rho}{\partial Q} = \int_{x=0}^{q} \int_{x=0}^{q} f(x, y) dx dy + \int_{x=0}^{\infty} \int_{y=0}^{\infty} f(x, y) dx dy + \int_{x=0}^{\infty} \int_{y=0}^{\infty} f(x, y) dx dy \]

Thus,

\[ \frac{\partial^2 \rho}{\partial q^2} = \int_{x=0}^{\infty} \int_{y=0}^{\infty} (r(x + y, q + Q, y) + (r + r) \int_{x=0}^{\infty} \int_{y=0}^{\infty} f(x, y, q + Q, x) dx dy + \int_{x=0}^{\infty} \int_{y=0}^{\infty} f(x, y, q + Q, x) dx dy ) dx dy \]

Therefore, \( \frac{\partial^2 \rho}{\partial q^2} \leq 0 \) for \( r > r + t \).

Also,

\[ \frac{\partial^2 \rho}{\partial q \partial Q} = \int_{x=0}^{\infty} \int_{y=0}^{\infty} \int_{x=0}^{\infty} \int_{y=0}^{\infty} f(x, y, q + Q, x) dx dy + \int_{x=0}^{\infty} \int_{y=0}^{\infty} \int_{x=0}^{\infty} \int_{y=0}^{\infty} f(x, y, q + Q, x) dx dy \]

Since \( \frac{\partial^2 \rho}{\partial q^2} - \frac{\partial^2 \rho}{\partial q \partial Q} - \frac{\partial^2 \rho}{\partial Q^2} \leq 0 \), it follows that \( \frac{\partial^2 \rho}{\partial q^2} - \left( \frac{\partial^2 \rho}{\partial q \partial Q} - \frac{\partial^2 \rho}{\partial Q^2} \right)^2 \geq 0 \). Hence, the expected profit is jointly concave in \( q \) and \( Q \).

**Proof of Proposition 4.** If we consider partial pooling with centralized control we need to distinguish between 7 possible cases. The total expected profit is

\[ \Pi = \int_{x=0}^{\infty} \int_{y=0}^{\infty} \left[ r(x + y, q + Q, x) - t(x + y, q - y, q) \right] dx dy + \int_{x=0}^{\infty} \int_{y=0}^{\infty} \left[ r(x + y, q + Q, x) - t(x + y, q - y, q) \right] dx dy \]

The first order conditions are

\[ \frac{\partial \Pi}{\partial q} = \int_{x=0}^{\infty} \int_{y=0}^{\infty} f(x, y, q + Q, x) dx dy + \int_{x=0}^{\infty} \int_{y=0}^{\infty} f(x, y, q + Q, x) dx dy - c_r \]

If we assume symmetric retailers we know that \( q = q = q \), i.e. the optimization problem reduces to a two-dimensional problem and a policy is denoted by \( (q, Q) \). We have:

\[ \frac{\partial \Pi}{\partial Q} = \int_{x=0}^{\infty} \int_{y=0}^{\infty} \left[ r(x + y, q + Q, x) - t(x + y, q - y, q) \right] dx dy + \int_{x=0}^{\infty} \int_{y=0}^{\infty} \left[ r(x + y, q + Q, x) - t(x + y, q - y, q) \right] dx dy - c_r \]

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\[ \frac{\partial^2\pi}{\partial q^2} = (t - r) \int_0^Q f(Q + q, x_i)dx_i + (t - r) \int_0^Q f(x_i, Q + 2q - x_i)dx_i, \]
\[ - r \int_0^Q f(q, x_i)dx_i - t \int_0^Q f(Q + q, x_i)dx_i \leq 0, \]
\[ \frac{\partial^2\pi}{\partial q^2} = \frac{\partial r}{\partial q} = (t - r) \int_0^Q f(Q + q, x_i)dx_i + (t - r) \int_0^Q f(x_i, Q + 2q - x_i)dx_i \leq 0, \]
\[ \frac{\partial^2\pi}{\partial q^2} = (t - r) \int_0^Q f(x_i, Q + 2q - x_i)dx_i + (t - r) \int_0^Q f(x_i, Q + q)dx_i, \]
\[ + (t - r) \int_0^Q f(Q + q, x_i)dx_i \leq 0. \]

From this the following inequalities hold:
\[ \frac{\partial^2\pi}{\partial q^2} \leq \frac{\partial r}{\partial q} \leq \frac{\partial^2\pi}{\partial q^2} \leq \frac{\partial^2\pi}{\partial q^2}. \]

Consequently, \( \frac{\partial^2\pi}{\partial q^2} - \left( \frac{\partial^2\pi}{\partial q^2} \right) \geq 0. \) Hence, the expected profit function is jointly concave in \( q, q, Q \) for symmetric retailers.

References