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Sample Autocorrelation Learning in a Capital Market Model

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Abstract

Adaptive agent models are supposed to result in the same limit behavior as models with perfectly rational agents. In this article we show that this claim cannot by accepted in general, even in a simple capital market model, where the agents apply sample autocorrelation learning to perform their forecasts. By applying this learning algorithm, the agents use sample means, the sample autocorrelation coefficient, and the sample variances of prices to predict the future prices, and to determine the demand for the risky asset. Therefore, even if the agents are not perfectly rational, we require that the agents' forecasts are consistent with the underlying information. In this article a sufficient condition for convergence is derived analytically, and checked by means of simulations. The price sequence as well as the sequence of parameters – estimated by means of sample autocorrelation learning – converge, if the initial value of the price sequence is sufficiently close to the steady-state equilibrium, and a random variable derived from the dividend process is not too volatile to skip the price trajectory out of the attracting region. Therefore, the market price can even diverge, and the region of convergence could become very small depending on the underlying parameters. Thus, divergence of the price sequences is not a pathological example, since it possibly occurs over a wide range of parameters. Therefore, the often claimed coincidence of adaptive agents models and ration agent models cannot be observed even in a simple capital market model.

JEL-Classification: D83, D84, G10.

Keywords: Artificial Markets, Bounded Rationality, Learning.

1 Introduction

Adaptive agent models are often supposed to converge to rational expectations models (REE), i.e. the limit behavior of the variables of the *adaptive agent model* is equal to the behavior the economic variables in the corresponding rational expectations model. If this were the case, the model is said to be *learned* by means of the underlying algorithm. Thus, if the REE would have been learned by an adaptive agent model, the REE concept will be supported by a learning mechanism furthermore, and the REE model serves as a benchmark in economic modeling.

Replacing a REE model by an adaptive agent model allows for infinitely many degrees of freedom to model the learning and the forecasting behavior of the agents. In principle the economist has to determine the underlying *learning algorithm* of the adaptive agent model, has to check whether the parameters of this particular algorithm converge, and whether convergence of the parameters results in coincidence of the limit behavior of the adaptive agent model and the REE model.

In this article the question of coincidence is discussed in detail within a simple capital market model. We shall demonstrate in the following that this claim cannot be accepted in general; even worse, the adaptive agent version and the REE version of this simple capital market model disagree over a wider range of parameters. Therefore, the adaptive agent model does not support the REE model in this particular setup, and there is know reason why this difference should disappear in even more complicated models.

As an alternative to rational expectations bounded rationality models have been proposed. They are based on behavioral foundations (See Sargent (1993), Arthur *et al.* (1997b)) or on *statistical mechanics* as in discrete choice models (See Brock and Hommes (1997) and Brock and Hommes (1998)). However – as already stated above – all models with non-rational agents allow for many degrees of freedom. For example the process of inference of new information can be modeled by means of linear rules, nonlinear techniques, genetic algorithms, or ad-hoc rules. Every forecast technique may result in different equilibrium behavior, if equilibria of the system exist. Although some models do well in explaining some stylized facts, some ad-hoc assumptions behind bounded rationality models can hardly be justified, at least from the REE point of view.

By using myopic agents using least squares learning rules Bray (1982), Blume *et al.* (1982), and Blume and Easley (1982) derived conditions for the system to converge to the rational expectations equilibrium. Rational (Bayesian) Learning and its convergence properties has been studied by Bray and Kreps (1987). Schönhofer (1997) discusses properties of least squares learning in great detail my means of simulations. In numerical examples he has demonstrated that least squares learning can result in complex dynamic behavior even if the underlying economic models seem to be relatively simple. Additionally, the estimated parameters are consistent with the agents' forecasting model even if the trajectories of the state variables of the model are chaotic. In Schönhofer (1997) *consistency* was defined in the sense of Sorger and Hommes (1998), such that the estimation error is zero in mean and uncorrelated.

In Timmermann (1994) the convergence of learning depends crucially on the prior information agents impose on the learning process. Thus, if agents try to learn about the long run dynamics without imposing strong prior information, learning cannot converge to REE. If, however, agents impose strong prior information and impose a unit root on prices and dividends of their model, i.e. agents confine their learning to the short run dynamics of the model, learning may converge to REE. Thus, models with adaptive agents may, but they need not converge to REE.

In Grandmont (1998) the problem of adaptive learning is analyzed more generally by the so called *general uncertainty principle*. In his seminal work Grandmont (1998) has demonstrated that if the agents do not only care about the positions of an economic system in equilibrium but also about the dynamics in a neighborhood of an equilibrium, the temporary dynamics of the system could be unstable. A closely related question to stability is the question whether perfect foresight equilibria can be attained, i.e. whether the future behavior of the trajectories of the state variables can be perfectly predicted by a given forecast rule. This problem has been investigated by Böhm and Wetzelburger (1997). In their article Böhm and Wetzelburger (1997) have shown that perfect foresight equilibria need not be attained even in deterministic systems. The question of attainability depends on the forecasting function and on the complexity of the law of motion of the system.

An analytical treatment of least squares learning is provided in Marcet and Sargent (1989). In their article agents use least squares learning to update their beliefs every period resulting in the agents' perceived law of motion; agents are not fully rational in the sense that they neglect their effect on the actual law of motion. The question arises whether the perceived law of motion and the actual law of motion can converge. In a linear setting, this problem could be solved by applying the stochastic approximation tools to the corresponding learning scheme, where the convergence properties of the learning scheme can be reduced to the convergence properties of an ordinary differential equation (See Ljung *et al.* (1992) and Kushner and Yin (1997)). For least squares learning this problem has been investigated by Marcet and Sargent (1989). Another application of stochastic approximation is by Chen and White (1998), where the corresponding algorithm for the periodical update of the vector of model parameters, can be given by neural networks, splines, or kernel functions. The authors called this method *nonparametric adaptive learning*. Stochastic approximation methods require that the underlying dynamic system – such as the vector of model parameters – is of the structure $x_t = x_{t-1} + \varepsilon_t g(t, x_{t-1})$, where ε_t has to become small (for regularity conditions on ε_t and $g(.)$ see Kushner and Clark (1978), Ljung *et al.* (1992), and Kushner and Yin (1997)). However, the following capital market model will not be of this structure.

In the following we want to highlight that even a simple adaptive agent model need not "converge" to its corresponding REE model. In this article we investigate the question whether sample autocorrelation learning results in convergence to the REE in a simple capital market model. Uncertainty enters via the stochastic dividend process. In section 2 the asset demand of the agents will be derived from maximizing mean-variance utility. Agents are allowed to differ only in their degree of risk aversion. By this assumption current demand for the risky asset is a function of the next period price, the next period dividend, and the variance of the portfolio. The REE derived in this model is a unique, and a stable steady-state equilibrium. Since the agents do not know the distribution of the dividends and the characteristics of the other agents, they have to perform predictions, i.e. the agents' forecasts are based on exogenous and endogenous variables of the underlying system. In contrast to the capital market models of Bray, and Bray and Savin considered in Marcet and Sargent (1989) the agents use past prices and past dividends to perform their predictions. The agents assume that the prices follow an AR(1) process, and the dividends are independent identically distributed. The parameters of the forecast models are derived by applying *sample autocorrelation learning* (SAC), as described in section 3. By using this algorithm the parameters are derived from calculating the mean of prices, the mean of dividends, and the first order autocorrelation coefficient of prices. The variance estimate is derived from the sample variances of prices and dividends. All estimates have to be based on the agents' current information, which consist of past prices and past dividends. In section 4 we present the *sufficient conditions for almost sure convergence* of the learning scheme to the rational expectations equilibrium. It will be shown that the initial price has to be sufficiently close to the steady-state REE to result in almost sure convergence. Additionally, a second condition requires that the realizations of the dividend process should not be too far away from its mean, such that the price process remains in its convergent region. The proof of this result is presented in appendix A. This result can be used to analyze the case where the mean and the variance of the dividend process are known, the case of almost sure convergence, and the case of prior information. Additionally, we use our result to investigate the problem of convergence for dividend processes on bounded support. Section 5 presents some numerical examples of convergent and divergent price sequences, and we check whether the analytical conditions of section 4 are weak or strong.

Therefore, we derive the result that the equilibrium dynamics of an adaptive agent models need not coincide with the dynamics of the corresponding REE model. As this article shows analytically and numerically, the sequence of state variables can diverge as well. This property of the capital market model is not restricted to a small region. Furthermore, the region where divergence occurs becomes quite large, if the interest rate is low, the number of the risky asset compared to the number of investors is high, and the degrees of risk aversion are high. Therefore, adaptive agent models and REE models need not agree in general.

2 The Stock Market Model

Agents, Wealth, and Asset Demand:

In this section we provide a brief description of the capital market model. Let us consider agents i, $i = 1, ..., n$ which are at time t able to invest their wealth w_t^i in a risky asset with price p_t and in a risk-free asset paying interest of r per unit of capital invested. The risk-free asset is the numeraire good. Every period the risky asset pays dividend d_t which is stochastic. According to these assumptions the budget constraint of agent i becomes:

$$w_t^i = (1+r)w_{t-1}^i + (p_t + d_t - (1+r)p_{t-1})\, q_{t-1}^i \ , \qquad (1)$$

where q_t^i is the amount of the risky asset held by agent i in period t. The model deviates from the REE assumptions such that the agents do not know the properties of the underlying dividend process (d_t), and attitude towards risk of the other agents. To derive their expectations, the agents infer from the price and dividend time series up to period

$t-1$, $(p_l)_{l=0}^{t-1}$ and $(d_t)_{l=0}^{t-1}$, respectively. Secondly, we assume that every period t the agents maximize mean-variance utility, i.e. they maximize

$$\mathbb{E}_{i,t}(w_{t+1}^i) - \frac{\zeta_i}{2}\mathrm{VAR}_{\,i,t}(w_{t+1}^i) \ , \tag{2}$$

where ζ_i measures the attitude towards risk of agent i. $\mathbb{E}_{i,t}(.)$ and $\mathrm{VAR}_{\,i,t}(.)$ are the agents' beliefs about the conditional expectation and the conditional variance of the wealth w_{t+1}^i (See Brock and Hommes (1997)). In section 3 these beliefs will be derived by the sample autocorrelation learning algorithm. The maximization of expected utility (2) such that the budget constraint (1) is satisfied results in the following demand function for the risky assent in period t:

$$q_t^{s,i} = \frac{(p_{t+1}+d_{t+1})^{e,i} - p_t(1+r)}{\zeta_i \sigma_i^2} \ , \tag{3}$$

where $p_{t+1}^{e,i}$, $d_{t+1}^{e,i}$, and σ_i^2 are the agents' beliefs of the conditional expected price of the next period, the conditional expected dividend, and the sum of the estimated conditional variance of prices and dividends, respectively. The information of our agents is given by past prices and dividends. From the agents' point of view, both variables are random, and have to be estimated by a forecasting rule. Nevertheless, the only way randomness enters into the capital market is the dividend process, but this is not known to the agents.

Remark 1 *Considering the budget constraint (1), the term* $\hat{R}_t = (p_{t+1}+d_{t+1})^{e,i} - p_t(1+r)$ *is often called excess return, since the return is adjusted by the return of an investment of* p_t *units of cash into the risk-free investment alternative. Therefore,* $p_t r$ *is deduced from the return to derive* $\hat{R}_t$*. Considering expected excess returns* $\hat{R}_t^e$*, such as in the numerator of equation (3),* $I\!P\left(\hat{R}_t^e > r\right) > 0$ *is necessary for risk averse agents to invest in the risky asset, and* $I\!P\left(\hat{R}_t^e \leq r\right) > 0$ *is necessary that risk averse agents invest in the risk-free security.*

Market Clearing, Price Dynamics, and REE:
The market clearing price will be derived from intersecting the horizontal sum of individuals' desired demands q_t^i with the supply of shares, which is fixed in our model at S. Thus, our market clearing condition becomes:

$$p_t = \sup\{p : \sum_{i=1}^{n} q_t^i = S\} \ , \tag{4}$$

The supply of the risky asset is fixed for every time t. Since the asset demand functions (3) are linear in p_t, equation (4) results in a unique market clearing price. By assuming homogeneity of the agents with respect to the estimates p_{t+1}^e, d_{t+1}^e, and σ^2, inserting the demand functions (3) into the market clearing condition (4), and perform some algebraic manipulations, we derive the market clearing price from (3) and (4):

$$p_t = \frac{1}{1+r}\left(p_{t+1}^e + d_{t+1}^e - \frac{S\sigma^2}{\sum_i \frac{1}{\zeta_i}}\right) \ . \tag{5}$$

Thus, given the price estimates $p^{e,i}_{t+1}$, the estimates of dividends $d^{e,i}_{t+1}$, and the estimated variance of prices and dividends σ_i^2, we are able to solve for p_t. This implies that the stock price can be described by a dynamical system that maps the agents estimates into the market clearing price p_t.

Since the agents do not know the properties of the dividend process and the preference ordering of the other agents, we assume that the agents behave like econometricians trying to interfere from their information set the behavior of the economic system by using linear forecast rules. More specific, we assume that the agents 'believe' in the following AR(1) forecast model, resulting in the perceived law of motion of prices:

$$\begin{pmatrix} p_t \\ d_t \end{pmatrix} = \begin{pmatrix} \alpha_p \\ \alpha_d \end{pmatrix} + \begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p_{t-1} - \alpha_p \\ d_{t-1} - \alpha_d \end{pmatrix} + \begin{pmatrix} \varepsilon_{p,t} \\ \varepsilon_{d,t} \end{pmatrix} , \tag{6}$$

where $\varepsilon_{p,t}$, $\varepsilon_{d,t}$ are assumed to be independent of each other and iid. with zero mean and conditional variance σ_p^2 and σ_d^2 respectively. According to (6) the agents belief in stationary price and dividend processes. Otherwise the above statistical model would not be plausible from the agents' point of view.

Remark 2 *Since the agents use past prices and dividends for their forecasts, inference is based on exogenous and endogenous variables.*

By the assumption of homogenous forecasts, our agents use the following model to predict the one period ahead price:

$$p_t^e = \alpha_p + \beta \left(p_{t-1} - \alpha_p \right) \ . \tag{7}$$

For the dividends, equation (6) results in the following forecast:

$$d_t^e = \alpha_d \ . \tag{8}$$

As already stated, the agents need the two step ahead forecast of prices and dividends in their demand functions (3), p^e_{t+1} and d^e_{t+1} respectively.

By the agents' assumption of independence of prices and dividends, the sum of the conditional variance of prices and the conditional variance of dividends σ^2 will be derived from the following equation:

$$\sigma^2 = \sigma_p^2 + \sigma_d^2 \ . \tag{9}$$

Within our setup the agents have the opportunity to forecast by means of a two-step-ahead forecast in equation (7), i.e.

Model 1

$$p^e_{t+1} = \alpha_p + \beta(\alpha_p + \beta(p_{t-1} - \alpha_p) - \alpha_p) = \alpha_p - \beta^2(p_{t-1} - \alpha_p) \ , \tag{10}$$

where the forecast is a fixed real number, or the agents "condition" on p^e_{t+1} on the unknown price p_t, i.e.

Model 2

$$p^e_{t+1} = \alpha_p - \beta(p_t - \alpha_p) \ , \tag{11}$$

where p^e_{t+1} is a function of p_t. Therefore, by defining $c := S/\sum_i(1/\zeta_i)$, and inserting either (10) or (11), and (8) into the market dynamics (5), we derive:

$$\begin{aligned} p_t &= \frac{1}{1+r}\left(p^e_{t+1} + d^e_{t+1} - \frac{S\sigma^2}{\sum_i \frac{1}{\zeta_i}}\right) \\ &= \frac{1}{1+r}\left(\alpha_p + \beta^2\left(p_{t-1} - \alpha_p\right) + \alpha_d - c(\sigma_p^2 + \sigma_d^2)\right) \ , \end{aligned} \tag{12}$$

for Model 1. For Model 2 we derive the following expression:

$$p_t = \frac{1}{1+r-\beta}\left(\alpha_p(1-\beta) + \alpha_d - c(\sigma_p^2 + \sigma_d^2)\right) \ . \tag{13}$$

If the dividend process $(d_t)_{t=0}^T$ follows an independent identically distributed process, with mean $\mu_d = \mathbb{E}(d_t)$ and variance $\sigma_d^2 = \mathrm{VAR}\,(d_t)$, the rational expectations equilibrium can be derived from (5):

$$p^* = \frac{1}{r}\left(\mu_d - \frac{S\sigma_d^2}{\sum_i \frac{1}{\zeta_i}}\right) \ . \tag{14}$$

The reader can easily check that this steady-state equilibrium is unique due to the linearity of the demand functions (3) in p_t, and the fact that σ^2 is equal to σ_d^2 in the steady-state of the system. Secondly, the steady-state REE is stable since the multiplier (eigenvalue) of the system $p_t = H(p^e_{t+1})$ derived by (5) is equal to $1/(1+r) < 1$ for $r > 0$.

3 Sample Autocorrelation Learning

Generally, the stock market allows different learning schemes to estimate the parameters of the forecast model (6), resulting in different predictions of the future prices and dividends. For example the agents use least squares learning (See Bray (1982), Marcet and Sargent (1989), Routledge (1996), and Schönhofer (1997)), sample autocorrelation learning, classifier systems (See Sargent (1993), Arthur *et al.* (1997b), and Arthur *et al.* (1997a)) or another adaptive learning rule.

The SAC Learning Algorithm:
By using sample autocorrelation learning the agents simply have to calculate the sample averages of the price $\bar{p}_t$

$$\bar{p}_t := \frac{1}{t}\sum_{\iota=0}^{t-1} p_\iota = \alpha_{p,t} \ , \tag{15}$$

where the sample mean of dividends $\bar{d}_t = \alpha_{d,t}$ is calculated in the same way. The first order autocovariance is derived from:

$$\mathrm{COV}_{\,t}(p_t, p_{t-1}) := \frac{1}{t-1}\sum_{\iota=1}^{t-1}\left(p_\iota - \bar{p}_t\right)\left(p_{\iota-1} - \bar{p}_{t-1}\right) \ . \tag{16}$$

The sample variance of prices is derived from:

$$\mathrm{VAR}_t(p_t) := \frac{1}{t}\sum_{\iota=1}^{t-1}(p_\iota - \bar{p}_t)^2 \ , \tag{17}$$

such that the conditional variance of prices is derived from:

$$\sigma^2_{p,t} := (1-\beta_t^2)\mathrm{VAR}_t(p_t) \ . \tag{18}$$

The coefficient β is derived from the first order autocorrelation coefficient:

$$\gamma_t := \frac{\mathrm{COV}_t(p_t, p_{t-1})}{\sqrt{\mathrm{VAR}_t(p_t)}\sqrt{\mathrm{VAR}_t(p_{t-1})}} = \beta_t \ . \tag{19}$$

It is worth noting that the autocorrelation coefficient stays within the interval $[-1, 1]$, because this property will be essential in the proof of Proposition 1. The variance $\mathrm{VAR}_t(p_{t-1})$ is equal to the variance of prices of the last period, i.e. $\mathrm{VAR}_t(p_{t-1}) = \mathrm{VAR}_{t-1}(p_t)$; the variance of dividends $\mathrm{VAR}_t(d_t) = \sigma^2_{d,t}$ is derived from (17) by using d_t for p_t. Therefore, we derive the estimates for α_p, α_d, β, and σ^2 by means of sample autocorrelation learning. These estimates can be used in the forecast models (7) and (8) to describe the agents' learning behavior.

Market Dynamics:

Since the agents do not know the demands of the other agents, the affine linear functions $p_t^{e,i} = \alpha_p + \beta(p_{t-1} - \alpha_p)$ are the perceived laws of motion of our agents, while the composition of demand functions (3), the forecasts of the agents, and the market clearing mechanism result in the implied law of motion of our capital market.

As already stated above, the parameters α_p, α_d, β, σ_p^2, and σ_d^2 are periodically updated by means of equations (15), (16), and (19), respectively. Therefore, the price dynamics are described by a dynamic system $p_t = F_1(p_{t-1}, \theta_t)$ for Model 1 and $p_t = F_2(\theta_t)$ for Model 2, where p_{t-1} is the price of the last period, and θ_t is the vector of estimated parameters $\theta_t := (\alpha_{p,t}, \alpha_{d,t}, \mathrm{COV}_t(p_t, p_{t-1}), \mathrm{VAR}_t(p_t), \mathrm{VAR}_t(d_t))'$ in period t.

Remark 3 *As stated in equations (15) to (19), the estimated parameters are functions of the observed prices $(p_\iota)_{iota=0}^{t-1}$, the observe dividends $(d_\iota)_{iota=0}^{t-1}$, and t. Therefore θ_t is a function of $d_0, ..., d_{t-1}, p_0, ..., p_{t-1}, t$.*

The sequence of prices (p_t) with sample autocorrelation learning for Model 1 is derived from equation (12), and the vector of estimates θ_t:

$$\begin{aligned} p_t &= \frac{1}{1+r}\Big(\alpha_{p,t} + \beta_t^2\,(p_{t-1} - \alpha_{p,t}) + \alpha_{d,t} - c(\sigma^2_{p,t} + \sigma^2_{d,t})\Big) \\ &=: F_1(p_{t-1}, \theta_t) \ , \end{aligned} \tag{20}$$

for the second model the price dynamics become:

$$p_t = \frac{1}{1+r-\beta}\Big(\alpha_p(1-\beta) + \alpha_d - c(\sigma_p^2 + \sigma_d^2)\Big) =: F_2(\theta_t) \ . \tag{21}$$

Equations (20) and (21) are the *implied (actual) laws of motion* of the capital market. The reader should note that until now we have not required that the dividend process is *iid.*. This should not be mixed up with the assumption on the agents, which believe in an *iid.* dividend process.

Remark 4 *The vector of parameters θ_t is derived by sample autocorrelation learning in Model 1 as well as in Model 2. Since the mappings $F_1(.)$ and $F_2(.)$ are not the same, the sequences of estimated parameters (θ_t) will become different. Despite the off-equilibrium behavior of both systems is different, the steady-states of both systems are the same (see equation (14)), as well as the limits of the (θ_t) for a convergent price sequence.*

Remark 5 *Since the sequence of prices follows the implied law of motion, (20) or (21) respectively, the limit of $\alpha_{p,t}$ is this asymptotic mean of the price sequence for a convergent sequence of prices, and β is equal to the first order autocorrelation coefficient of prices, this setup satisfies the conditions of a consistent expectations equilbirium (CEE) in the terminology of Sorger and Hommes (1998). The goal of the CEE concept is to provide an equilibrium concept, where the requirements with respect to information and the analytical abilities of the agents are decreased enormously compared to perfectly rational agents. Nevertheless the CEE concept requires consistency of the agents' forecasts with the underlying data. Therefore, if $(p_t) \to p^*$ the REE as well as the CEE are "learned" in this capital market model.*

4 Convergence of SAC-Learning

In section 2 we have described the dynamics of the price sequences by (20) and (21), for Models 1 and 2 respectively, and the way the vector of parameters θ_t is estimated. Now we investigate the question, whether the price sequence (p_t) converges to the rational expectations equilibrium. Within this section we highlight our result that even in a simple capital market model, which is linear in p_{t-1} for a fixed vector of parameters θ, and a simple learning rule, the adaptive agent model need not converge to the rational expectations equilibrium. In appendix A a sufficient condition for convergence is derived. This condition is sufficient for Model 1, where $p_t = F_1(p_{t-1}, \theta_t)$ as well as for Model 2, where $p_t = F_2(\theta_t)$. Despite we derive a sufficient condition for convergence, we want to emphasize that the price sequence of the adaptive agent model diverges, i.e. the REE-steady-state equilibrium is not attained even in a simple model. This section will be organized as follows: First we present our main result. Secondly, we use our main result to analyze the following cases:

1. α_d and σ_d^2 are common knowledge.
2. Almost sure Convergence – Convergence to the REE.
3. Prior information on a stochastic dividend process.
4. The dividend process has bounded support, i.e. $d_t \in [d_l, d_h]$.

Before, we present our main result, let us define a real valued sequence

$$z_t := \alpha_{d,t} - c\sigma^2_{d,t} \ . \tag{22}$$

This sequence may be either deterministic or stochastic. If the sequence converges, the limit of this sequence will be denoted by z.

Proposition 1 *The sequence of prices (p_t) derived from (20) or (21) converges, if the sequence (z_t) converges to z, and the following conditions are met:*

$$|p_0 - \frac{z}{r}| \leq \nu \leq \frac{r}{c} \ , \tag{23}$$

and

$$\sup_t |z_t - z| \leq r\nu - c\nu^2 \ . \tag{24}$$

As stated above the price sequence converges, if the initial price p_0 is sufficiently close to $p^* = z/r$, and if the path of the dividend process (d_t) – which may be either stochastic or deterministic – does not skip out the sequence (z_t) of the convergent region (24). Therefore, an initial price which is very close to p^* is not sufficient for convergence. The system can always be disturbed by the dividend process, such that inequality (24) is not satisfied. The probability that inequality (24) is not met depends crucially on the law of the dividend process.

Remark 6 *The market clearing mechanism (4) has been defined without any lower bound for the price sequence, for example $p_t \geq K_l = 0$. One might suppose that a lower bound of the prices sequence K_l would be sufficient for convergence to the REE price p^*, derived from equation (14). Proposition 1 contradicts this claim. Even if the price sequence is bounded from below, convergence need not take place within the capital market model. If $p_t \geq K_l$ and prices do not converge, then $p_t \to K_l$ instead of $p_t \to -\infty$. We also checked this result by means of simulations (see section 5). Although the prices are bounded from below – $p_t \geq 0$ in our numerical examples – we derived price sequences "diverging" to the lower bound, despite the steady-state equilibrium is a positive real number, and prices are bounded from below.*

I. α_d and σ_d^2 are common knowledge:
Let us suppose that the agents are well informed about the prosperities of the process (d_t), i.e. the agents know the expectation and the variance of the process. The agents' beliefs that the dividend process is *iid.* results in $\alpha_d = \alpha_{d,t} = \mathbb{E}(d_t)$ and $\sigma_d = \sigma_{d,t} = \text{VAR}\,(d_t)$. Then z_t is equal to z for all periods t and all agents i. In this case the capital market model becomes a deterministic model. Considering Proposition 1, condition (24) will be always met, however condition (23) demands for an initial price p_0 satisfying $|p_0 - \frac{z}{r}| \leq \frac{r}{c}$. Since this condition is only sufficient, the reader might suppose that the deterministic system converges as well if condition (23) will not be met. Nevertheless, checking the eigenvalues (multipliers) of the system (12) and the system (13) results in eigenvalues outside unit circle, which indicate that the system need not be stable. Secondly, a numerical analysis of systems (12) and (13) shows that the some price sequence diverge even in a deterministic

system. However, the initial price may be allowed to be located ten to twenty times off the interval required by condition (23) where the price sequence still converges, which is relatively high compared to the results in stochastic dividend setups presented in section 5.

II. Almost Sure Convergence – Convergence to the REE:
From Proposition 1 we deduce for a stochastic sequence (z_t):

Corollary 1 *[***Sufficient Condition for a.s. Convergence:***] The sequence of prices (p_t) derived from (20) or (21) converges with probability one, if the sequence (z_t) converges to z with probability one, i.e. $\mathbb{P}(\lim_{t\to\infty} z_t = z) = 1$, and the following conditions are met:*

$$|p_0 - \frac{z}{r}| \le \nu \le \frac{r}{c} \ , \tag{25}$$

and

$$\sup_t |z_t - z| \le r\nu - c\nu^2 \ . \tag{26}$$

Remark 7 *If condition (23) holds, the prices (p_t) converge for all paths (z_t) satisfying (24). The conditions of Proposition 1 do not contain any statement involving the stochastics of (z_t). Whether (p_t) converges almost surely to the REE is simply the problem whether the set of paths satisfying (24) has full probability. This is the case for certain distributions (of d_t) with bounded support.*

Example 1 (Iid. Dividends:) *An iid. dividend process is only a special case of Corollary 1. If the z_t derived from the dividend process stay within the interval given by (26) with probability one, and (25) is met, then the sequence of prices converges to the REE. Iid. dividends – especially (iid.) normally distributed dividends – are often used in economic models. In this case the REE is a constant as stated in equation (14). Therefore this special case serves as a benchmark, since the REE is of a very simple structure. Therefore, if one particular path of the dividend process results in a sequence (z_t) which does not satisfy condition (24) the price sequence need not converge. For normally distributed dividends the probability that (26) is met for all sample paths is zero. Therefore, the system need not converge even with iid. normally distributed dividends.*

Thus, by means of proposition 1 we derive the result that REE and adaptive agent models need not agree in the limit, even if the model and the stochastic properties of the dividend process are relatively simple. The numerical results of section 5 will show that although our conditions are only sufficient, divergence takes place, if we leave the bounds derived above for approximately more than two times.

III. Prior Information:
Corollary 1 treats the case where a (z_t) converging to its limit z satisfies (26) with probability one. However, $|z_t - z| \le \mu \le r\nu - c\nu^2$ is not fulfilled with probability one in general. Nevertheless, let us suppose that the agents have some prior information about

the process (z_t), i.e. the agents observe a sequence $(d_{t-l})_{l=1}^{H}$ of H dividend payments before trade starts at $t = 1$. This assumption can be motivated by some prior information of the investors on the firms' profits before the asset is traded on a capital market. Therefore, let us define the sequence (z_t^H) derived by means of (22) from the sequence of the observed dividend payments $(d_{t-l})_{l=1}^{H}$. If $(z_t^H)_{t=1}^{\infty}$ satisfies condition (24) due to prior information, i.e. $\lim_{H\to\infty} \mathbb{P}((\sup_t |z_t^H - z| - \mu) \geq \varepsilon) = 0$, and if the initial price p_0 meets (23), we derive the result $\mathbb{P}(|p_t - p^*| \geq \varepsilon) \leq \lim_{H\to\infty} \mathbb{P}(\sup_t |z_t^H - z| \geq \mu) = 0$. This result is summarized in the following corollary:

Corollary 2 *Let H be the number of periods the agents observe the sequence of cash flows (d_t) before trade starts, i.e. $d_0, ..., d_{H-1}$. If the initial price p_0 satisfies condition (23), $z_t \to z$, and $\lim_{H\to\infty} \mathbb{P}(\sup_t |z_t^H - z| - \mu \geq \varepsilon) = 0$, for all t, then the sequence of prices (p_t) converges to its limit p^* (in probability).*

Remark 8 *Generally, the set of paths not satisfying (24) has positive probability. In this situation, it seems relevant to comment on the assumptions. Is it realistic to assume (23) and (24) for instance in a situation, where p_0 is an estimate of z/r based on prior information involving $d_{-H+1}, \ldots, d_0$? Moreover, z_t^H is based on $d_{-H+1}, \ldots, d_t$ and varies less than an estimate of $\alpha_d - c\sigma_d^2$ based on $d_0, \ldots, d_t$ only.*
Corollary 2 considers the problem whether the probability – that (23) and (24) are satisfied – converges with increasing H to 1. If this is the case, then, even if prices do not converge a.e., agents will observe a path of convergent prices with high probability.

IV. Dividend Process on bounded Support

Let us consider a dividend process on bounded support, i.e. $d_t \in [d_l, d_h]$. The length of the interval $[d_h - d_l]$ will be called λ in the further analysis. Now, the maximal difference of $|z_t - z|$ can be obtain as a function of d_l, d_h, and the model parameters as presented in Appendix B. In the case of a bounded support of the dividend process the following result can be derived from Proposition 1:

Corollary 3 *If $c\lambda \leq 1$, $|p_0 - \frac{z}{r}| \leq \frac{r}{2c}$, and $\max\{|z - d_l|, |z - d_h|\} \leq \frac{r^2}{4c}$, then the condition (24) of Proposition 1 is fulfilled.*
Therefore, if $c \leq \min\{\frac{r^2}{4|z-d_l|}, \frac{r^2}{4|z-d_h|}, \frac{1}{\lambda}\}$ and $|p_0 - \frac{z}{r}| \leq rc$, then the conditions (23) and (24) are satisfied.

This implies that convergence takes place if the number of the risky asset S is low, the number of agents N is high, the degree of risk aversion is low, the interest rate r is high, the differences $|z - d_h|$ and $|z - d_h|$ are not too large, and the length of the support of the dividend process $d_h - d_l$ is small.

Remark 9 *The degree of skewness can be defined by a parameter $\psi \in [0, 1]$, where ψ satisfies the equation $\mu_d = \psi d_l + (1 - \psi)d_h$. If $\psi > 0.5$ the distribution is said to be skewed to the right. Therefore, using $|z - d_l| \leq |\mu_d - d_l|$, the conditions on c of Corollary 3 become $c \leq \min\{\frac{r^2}{4(1-\psi)\lambda}, \frac{r^2}{4|z-d_h|}, \frac{1}{\lambda}\}$. Thus, for any given interest rate r and an interval with length λ, the coefficient c where condition (24) is met increases with the degree of skewness.*

Example 2 *[Uniform distribution:] Let us suppose that* d_t *is uniformly distributed on the interval* $[d_l, d_h]$*, (23) is satisfied, and* $c(d_h - d_l) < 1$*. Then, according to Corollary 3 and Remark 9 almost sure convergence takes place if* $c \leq \min\{\frac{r^2}{2(d_h-d_l)}, \frac{r^2}{4|\frac{6\lambda - c\lambda^2}{12} - d_h|}, \frac{1}{d_h - d_l}\}$*, where (23) has to be satisfied as well. For* d_t *uniformly distributed on the unit interval, this yields* $c \leq \min\{\frac{r^2}{2}, 1\}$*.*

Corollary 4 *If* $c\lambda \geq 1$*,* $|z - d_h| \leq \frac{r^2}{4c}$*, and* $r^2 \geq \frac{4(z-d_l)}{\lambda}$*, then the condition (24) of Proposition 1 is fulfilled.*

Proof See Appendix B. ■

Example 3 *[Uniform distribution:] Let us suppose that* d_t *is uniformly distributed on the unit interval, (23) is met, and* $c > 1$*. The reader can easily verify that the conditions of Corollary 4 result in* $c \in [\max(1, 6 - 3r^2), -3 + \sqrt{9 + 3r^2}]$*. Therefore, if* $c \in [\max(1, 6 - 3r^2), -3 + \sqrt{9 + 3r^2}]$ *and* p_0 *is sufficiently close to the steady-state, then the convergence of the price sequence takes place.*

Last but not least, the prove of convergence did not require the assumption that β is estimated by means of the first order sample autocorrelation coefficient (see appendix A). Therefore, we can replace β_t by a sequence of (ϱ_t) that satisfies $-1 \leq \varrho_t \leq 1$. Under this assumption the above convergence results continue to hold. This results in the following corollary:

Corollary 5 *If the forecast rule (7) is replaced by:*

$$p_t^e = \alpha_p + \varrho_t (p_{t-1} - \alpha_p) \ , \tag{27}$$

and the sequence of (ϱ_t) *fulfills* $-1 \leq \varrho_t \leq 1$*. Then the conditions of Proposition 1 are sufficient for convergence of* (p_t)*, and the sequence of parameters* $\left(\hat{\theta}_t := \left(\alpha_{p,t}, \alpha_{d,t}, \sigma^2_{p,t}, \sigma^2_{d,t}\right)'\right)$ *converges to its limit* $\hat{\theta}$ *with probability one.*

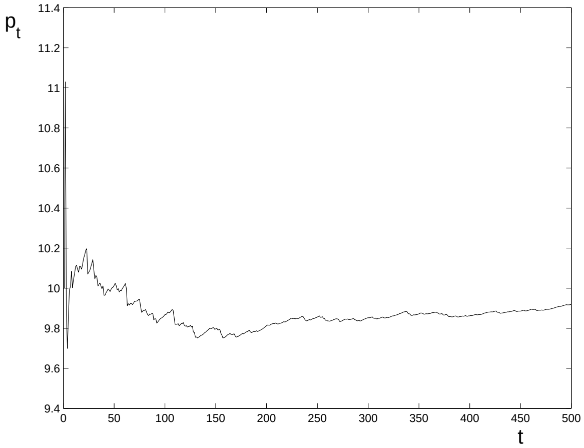
Corollary 6 *This result can be generalized even further by assuming a forecast function* $p_t^e = g(p_{t-1}, ..., p_0)$*, where* $g(.)$ *is bounded by* K^g*. Then the forecasts become* $p_{t+1}^e = g(g(p_{t-1}, ..., p_0), p_{t-1}, ..., p_0) := g_1(.)$ *and* $p_{t+1}^e = g(p_t, p_{t-1}, ..., p_0) := g_2(.)$*, for Model 1 and Model 2 respectively. The Lemmas of appendix A continue to hold, if the term* $\{\alpha_{p,t} \vee p_{t-1}\}$ *is replaced by* $\{g_j(.) \vee K^g\}$*, for* $j = \{1, 2\}$*.*

From the above results we are able to conclude that the adaptive agent model need not converge to the rational expectations equilibrium. Additionally, the regions where the sufficient conditions for convergence of Proposition 1 are met will become small, if the interest rate is low, the number of the risky asset compared to the number of investors is high, and the degrees of risk aversion are high. Therefore, the claim that the limit of an adaptive agent model and rational expectations models have the same dynamics, cannot be supported by the above results. There is neither theoretical nor empirical evidence that the initial values of prices and the paths of the dividend process are within the bounds for convergence derived in Proposition 1. As stated above, these regions become

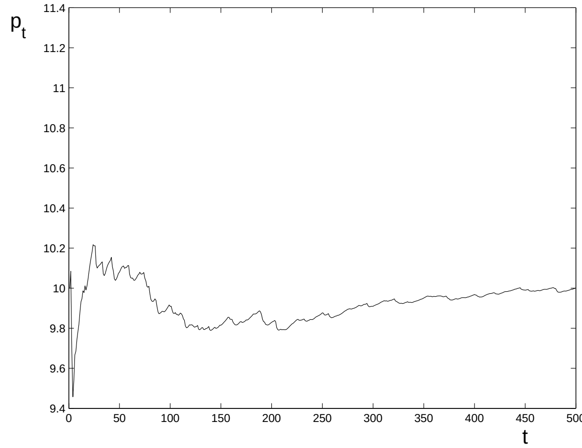
very small with low interest rates r, and a high coefficient c. Thus, we have derived a counterexample for the claim that adaptive agent models converge to their corresponding REE model. Additionally, we have to keep in mind that this result of divergence already occurs in simple capital market model with a simple learning rule.
Nevertheless, our condition is only sufficient for convergence. In section 5 we shall present examples, where (p_t) converges, despite the sequence (z_t) and/or the initial price p_0 do not met the sufficient conditions for convergence. However, even a high percentage of price sequences diverge if our conditions are not met. The number of divergent sequences increases, the more we depart from the regions derived in Proposition 1.

5 Simulations

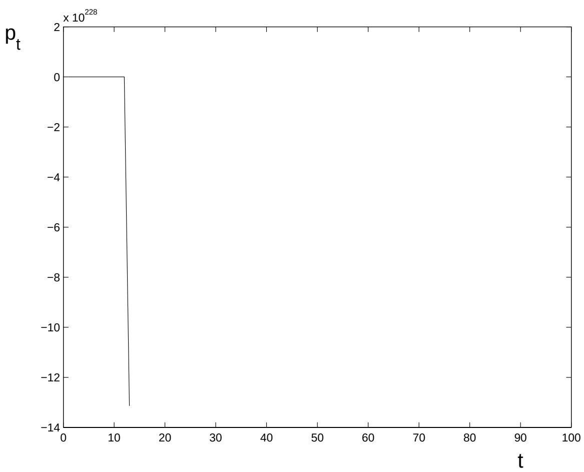
In this section we present some examples of convergent and divergent price sequences, and check the conditions derived in Proposition 1 numerically. Despite we cannot verify a.s. convergence or convergence in probability numerically, we could check whether the generated price sequences are near the steady-state equilibrium after a certain number of periods. Within this section we are very generous with the term convergence, with the purpose to be on the "save side", since we want to show that divergence can occur if the conditions of Proposition 1 are not met. Therefore, we consider a price sequence generated by means of the computer to be divergent, if $|p_t - z/r|$ is out of the range of the computer ($\pm 10^{309}$ in MATLAB 5.2), or at least more than $p^* \pm 10^5$ within 500 periods. This assumption results from our observations: If a particular price sequence results in prices such that $|p_t - p^*| \geq 10^4$ within the first 500 time steps, the simulated price sequence becomes soon extremely negative, i.e. $p_t \ll -10^{100}$. To remain on the save side we have chosen $|p_t - p^*| \geq 10^5$. For all simulation runs we used MATLAB 5.2.
Figure 1 presents the development of six price sequences in the first 100 or 500 periods. The dividend process (d_t) follows an iid. normally distributed process, i.e. $d_t \sim N(\mu_d, \sigma_d^2)$. On the left-hand side figure 1 presents simulation runs of Model 1, while the right-hand side shows three paths generated with Model 2. The subplots (a) and (b) of figure 1 show convergent paths, while subplots (c-f) present divergent paths. In (c) and (d) the price sequences have no lower bound, while in the (e) and (f), prices are bounded from below, such that $p_t \geq 0$. As already stated in Remark 6, price sequences need not converge if $p_t \geq K_l$.
First, we checked the convergence properties of prices if the sequence (z_t) is uniformly distributed on $\mathcal{A} := [\mu_z - k\kappa, \mu_z + k\kappa]$, where $\kappa := r\nu - c\nu^2$. For each $\tilde{\nu} = p_0 - z/r = \pm 0.5, \pm 0.7$, and $\pm .9$, with $\nu = |\tilde{\nu}|$, we performed 200 runs with 1000 time steps. In all runs we set $c = 1$, and $r = 1$. The process (z_t) follows an iid. and uniform distribution with $\mu_z = 1$, where the support of d_t is $\mathcal{A} := [\mu_z - k\kappa, \mu_z + k\kappa]$. The results of the simulation runs are presented in table 1. Since Proposition 1 provides only a sufficient condition for convergence, a lot of the price series converge for $k > 1$ as well. For $k = 1$ all sequences converge. However, convergence properties deteriorate as k increases. For $|z_t| \leq 5\kappa$, we observed no divergent price paths. However, the convergence properties change drastically, if k is further increased. Especially, in Model 1 the price sequences diverge for $k \geq 10$. Model 2 exhibits convergent paths even if $k = 10$. However, if $k \geq 12$, we derived no convergent sequences.



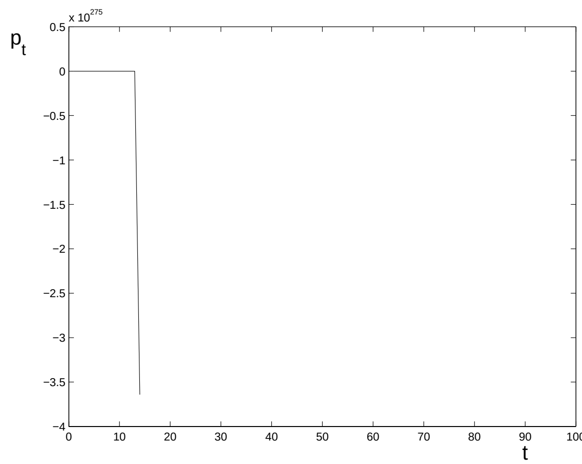
(a) `Model 1:` $d_t \sim N(11,1)$
$c=1$, $r=1$, $p_0=10$



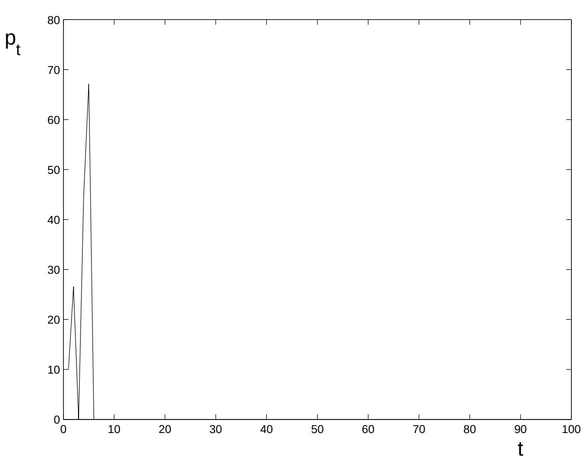
(b) `Model 2:` $d_t \sim N(11,1)$
$c=1$, $r=1$, $p_0=10$



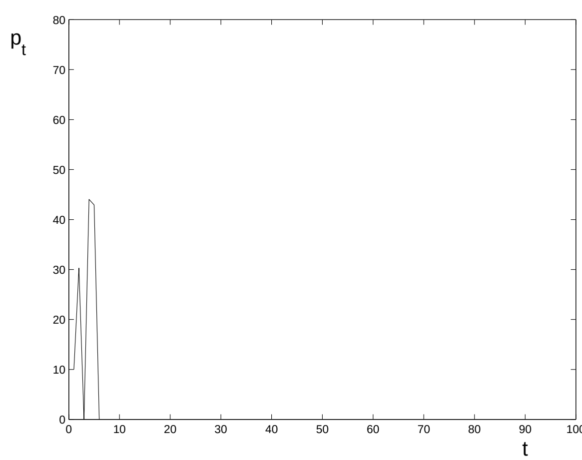
(c) `Model 1:` $d_t \sim N(60,50)$
$c=1$, $r=1$, $p_0=10$



(d) `Model 2:` $d_t \sim N(60,50)$
$c=1$, $r=1$, $p_0=10$



(e) `Model 1:` $d_t \sim N(60,50)$,
$p_t \geq 0$, $c=1$, $r=1$, $p_0=10$



(f) `Model 2:` $d_t \sim N(60,50)$,
$p_t \geq 0$, $c=1$, $r=1$, $p_0=10$

Figure 1: Convergent an divergent price sequences

k	Divergent [%]	
	Model 1	Model 2
2	0.0	0.0
5	0.0	0.0
8	22.0	0.0
9	98.0	40.0
10	100.0	97.5
12	100.0	100.0
20	100.0	100.0

Table 1: Convergence of (p_t) with (z_t) on $\mathcal{A}$.

Next, we checked the convergence properties pathwise, i.e. are there any sample paths where the conditions of Proposition 1 are not met, and the price path converges to the steady state equilibrium, for Model 1 and Model 2 respectively. This is done in the following way: We set $\mu_d = 10$, $r = 1$, $c = 1$, $\nu = 0.5$, and $\sigma_d^2 = 10$ such that $p^* = z = 0$, and $\kappa = 1/4$. Dividends are considered to be normally distributed. We derived simulation runs, where the sequences (z_t) do not satisfy $\sup_t |z_t| \leq \kappa$ and prices converge. However, if we consider the averages of the sequence (z_t) in the first 10, 20, 50, and 100 time steps, we observe that prices tend to diverge, if the mean of (z_t) is far above or below $z = 0$. For convergent sample paths the sequence (z_t) need not satisfy $|z_t| \leq \kappa$, nevertheless their mean has to lie approximately within the interval $(-3\kappa, 3\kappa)$. Only 8% of the price sequences show a convergent trend if the mean of the $(z_t)_{t=1}^{50}$ is outside $(-3\kappa, 3\kappa)$. Secondly, we observe that the earlier z_t is far above or below z (i.e. t is small), the stronger the effect on convergence or divergence of the price sequence. This effect is already excepted from the analytical point of view, because the weights of each observation of p_t and d_t decrease in t, since the parameters of the models are derived by calculating the mean values and the variances. These results are observed for Model 1 as well as Model 2.
Additionally, we generated normally distributed dividends such that inequality (24) is fulfilled, i.e. we normalized the sequence (z_t). Additionally the initial condition is met, i.e. $|p_0 - z/r| \leq \nu$. In all these runs the price sequences converge as we have already observed with the uniformly distributed dividends. Next, we started with $p_0 - z/r \leq k\tilde{\nu}$, and compress (z_t) to the region $|z_t - z| \leq k\,(r\nu - c\nu^2)$. In contrast to the simulation runs with the uniformly distributed z_t, there is a least one z_t which satisfies $|z_t - z| = k\kappa$. Therefore, we definitely know that there is a z_t on k times the bound of Proposition 1. By means of these simulation runs we wanted to see whether conditions of Proposition 1 are strong or weak. Therefore, we set $p_0 = z/r + \tilde{\nu}$, such that $|p_0 - z/r| \leq \nu \leq r/c$, generated paths of the dividends process (d_t) with $d_t \sim N(\mu_d, \sigma_d^2)$ resulting in the sequence (z_t), compressed (z_t) such that $|z_t - z| \leq k\kappa$ with $k \in [1, 10]$, and checked whether prices converge. For each vale of k we performed 50 simulation runs for $\tilde{\nu} = \pm 0.5$, ± 0.8, ± 0.9, and ± 1 with 250 time steps (for $\tilde{\nu} = \pm 1$ we have $\kappa = 0$). The parameters where set to $\mu_d = 10$, $r = 1$, $c = 1$, and $\sigma_d^2 = 10$. The results of these simulations are presented in table 2.

	Model 1	Model 2
k	Divergent [%]	Divergent [%]
1.0	0.0	0.0
3.5	0.0	0.0
4.0	3.8	3.8
5.0	32.2	30.4
10.0	90.7	91.1

Table 2: Convergence with compressed (z_t).

We conclude from table 2 that the price sequence is almost convergent, if the analytical bounds will be multiplied by a factor less than $k = 3.5$. As k increases the percentage of convergent sequences decreases. For $k = 4$ the increase in the percentage of divergent sequences is relatively low. However, as k increases further to a factor of approximately $k = 5$, approximately less than 70% of the sequences converge. For $k > 5$ the number of convergent sequences decreases rapidly. For $k > 10$ only a view sequences converge.
By a careful look at the price paths, we could see that only the sequences starting close to the steady-state converge. With $p_0 - z/r = \pm 0.5k$ and $p_0 - z/r = \pm 0.7k$, 33% and 9%, and 32% and 8% of the price sequences converge for $k = 10$ in Model 1 and Model 2, respectively. Such as for $k = 10$, convergent sequences mostly occur for small ν for $k = 5$. We observe no convergent price sequences for $k > 12$ in both models.

6 Conclusions

In this article we investigated whether the steady-state rational expectations equilibrium can be learned by means of sample autocorrelation learning. The capital market consists of myopic agents maximizing mean-variance utility. Agents only differ with respect to their attitude to risk. Without knowledge of the distribution of the dividend process, and the characteristics of the other agents, as assumed in REE models, the agents have to perform forecasts of the future prices and dividends to derive their demand schedule. In this model these forecasts are performed by means of a linear model, where the coefficients are estimated by means of sample autocorrelation learning. Using this algorithm the agents simply have to calculate the sample means, the covariance of prices, and the variances of prices and dividends. Therefore, the agents' forecasts are consistent with the data observed, given the underlying statistical model.
The question arises whether the vector of these parameters converges with probability one. In this article this question is solved analytically and checked by means of simulations. First of all, the parameters converge if the sequence of prices converges. In this article we have shown that (i) if the initial price p_0 is not too far away from the steady-state equilibrium, and (ii) that the supremum of the sequence (z_t) – defined by the sum of the sample mean of dividends plus the product of the sample variance of dividends times a factor consisting of stock supply and the degrees of risk aversion – is not too far away from the limit of this sequence are sufficient for convergence. Additionally, we provide sufficient

conditions for convergence if the dividend process has bounded support. It is worth noting that divergence can occur as well, if the price sequence is bounded from below, for example if prices are restricted to be non-negative. As we have checked by means of simulations our analytical result is only a sufficient condition for almost sure convergence. If the sequence (z_t) still stays within four times the region derived analytically, the price sequence remains convergent in almost all simulation runs.

Nevertheless, the main result of this article is the following: It is often claimed that adaptive agent models "converge" to a corresponding rational expectations model, i.e. after the parameters of the adaptive agent model have converged (where convergence is supposed as well), the dynamics of the adaptive agent model and the corresponding REE model are the same. In this article we have shown that this need not be the case, since we have derived examples where the sequences of prices diverge even if the REE is a positive real number. Next, we know from the analytical part and the simulation results, that the initial value of prices and a random variable derived from the stochastic dividend process, have to fulfill the conditions of section 4 or should not be too far from the bounds derived analytically to attain convergence. Additionally, the regions where the conditions for convergence are met will become small, if the interest rate is low, the number of the risky asset compared to the number of investors is high, and the degrees of risk aversion are high. Therefore, we have shown that almost a simple model – such as this capital market model – need not converge to the REE equilibrium. The region where the REE model and the adaptive agent model coincide, could become very small depending on the exogenous variables of the model. Thus, the claim that adaptive agent models and REE models agree, or at most agree over a wide range, cannot be accepted in general.

A Proof of Convergence

In this appendix we investigate the convergence of the capital market models 1 and 2. The process of the market price p_t is either described by the mappings (20) or (21), and the sample autocorrelation learning algorithm as stated in section 3. In this appendix we derive sufficient conditions for convergence for model 1. By replacing (20) by the mapping (21), the reader can easily check that the following lemmas and the condition for convergence remain valid for model 2 as well.

Let us assume p_0, r, and $c \in I\!R$. From the economic point of view we assume $r > 0$ (*positive interest payments of the riskfree security*) and $c > 0$ (*risk averse agents*). Furthermore, we assume that the sequence $(z_t)_{i=0}^{\infty}$ converges to its limit z. This does not require that the dividend process is iid.

In the next step we centralize the price sequence (p_t). If (z_t) is convergent with probability one, and (p_t) follows equation (20), then the transformed sequence $\left(p_t + \frac{z}{r}\right)$ satisfies (20) for $(z_t + z)$. Therefore, without loss of generality we can assume that (z_t) is a sequence converging to zero. Since, the sequence of parameters (θ_t) converges if (p_t) converges to the steady-state equilibrium, the analysis of convergence of the sequence (θ_t) can be reduced to the question of whether (p_t) converges. In a steady-state equilibrium p^* we have $\alpha_p = p^*$, $\alpha_d = \mu_d$, $\mathrm{COV}\,(p_t, p_{t-1}) = 0$, $\sigma_p^2 = \mathrm{VAR}\,(p_t) = 0$, and $\sigma_d^2 = \mathrm{VAR}\,(d_t)$. By the above transformation, the question of convergence of prices results in whether $\lim_t p_t \to 0$. First of all we show that there exists an upper bound for the sequence of prices:

Lemma 1 *The sequence* (p_t) *is bounded from above and* $\limsup_t p_t \leq 0$.

Proof Let $p_t^{max} = max\{p_i : i \leq t\}$. For any t such that $p_t = p_t^{max}$ we have:

$$\begin{aligned} p_t^{max} = p_t &= \frac{1}{1+r}\left(z_t - c\sigma_{p,t}^2 + (1-\beta_t^2)\alpha_p + \beta_t^2 p_{t-1}\right) \\ &\leq \frac{1}{1+r}\left(z_t + (1-\beta_t^2)p_t^{max} + \beta_t^2 p_t^{max}\right) \\ &= \frac{1}{1+r}\left(z_t + p_t^{max}\right) \ . \end{aligned} \tag{28}$$

The first line in (28) is given by (20). As assumed in section 3, $\alpha_{p,t}$ is the mean of prices at t. Therefore, $\alpha_{p,t} \leq p_t^{max}$ by the definition of p_t^{max}. This consideration and fact that $c\sigma_{p,t}^2 \geq 0$ result in the third line of (28). Therefore, by inequality (28) the sequence of prices is bounded by:

$$p_t^{max} \leq \frac{z_t}{r} < \infty \ . \tag{29}$$

Therefore p_t is bounded from above; particularly, $\limsup_t p_t$ is finite. Since $\limsup_t \alpha_{p,t} \leq \limsup_t p_t$, we derive from (20):

$$\limsup_t p_t \quad \leq \quad \frac{1}{1+r}\left(\limsup_t z_t + \limsup_t \left((1-\beta_t)^2\alpha_{p,t} + \beta_t^2 p_{t-1}\right)\right)$$

$$
\begin{aligned}
&\leq \frac{1}{1+r}\left(\limsup_t z_t + \limsup_t \left(\alpha_{p,t} \vee p_{t-1}\right)\right) \\
&\leq \frac{1}{1+r}\left(0 + \limsup_t p_t\right) \ , \qquad (30)
\end{aligned}
$$

which results in $\limsup_t p_t \leq 0$ by inequality (30), where $\alpha_p \vee p_{t-1} := \max\left(\alpha_{p,t}, p_{t-1}\right)$. ■

In the second step we define a sequence (u_t), and show that this sequence converges, if $r > 0$ and $z_t \to 0$.

Lemma 2 *Let $r > 0$, (z_t) a sequence of reals with $z_t \to 0$, and let u_t be defined by:*

$$
u_t = \frac{1}{1+r}\left(z_t + \alpha_{u,t} \vee u_{t-1}\right) \ , \qquad (31)
$$

where $\alpha_{u,t} = \frac{1}{t}\sum_{i=0}^{t-1} u_i$. Then $u_t \to 0$.

Proof $\limsup_t u_t \leq 0$ may be proved exactly as the corresponding statement for (p_t) in Lemma 1. To prove $u_t \to 0$, we have to show that $\liminf_t u_t \geq 0$. From equation (31) we derive:

$$
\begin{aligned}
u_t &= \frac{1}{1+r}\left(z_t + \alpha_{u,t} \vee u_{t-1}\right) \\
&\geq \frac{1}{1+r}\left(z_t + u_{t-1}\right) \ . \qquad (32)
\end{aligned}
$$

Next, recursive substitution in (32) yields:

$$
\begin{aligned}
u_t &\geq \frac{1}{1+r} z_t + \left(\frac{1}{1+r}\right)\left(\frac{1}{1+r} z_{t-1} + u_{t-2}\right) \\
&\vdots \\
&\geq \frac{1}{1+r} z_t + \left(\frac{1}{1+r}\right)^2 z_{t-1} + \ldots \\
&+ \left(\frac{1}{1+r}\right)^{t-1} z_{t-2} + \left(\frac{1}{1+r}\right)^t z_1 + \left(\frac{1}{1+r}\right)^t u_0 \\
&= \sum_{i=1}^{t} z_i \left(\frac{1}{1+r}\right)^{(t-i+1)} + u_0 \left(\frac{1}{1+r}\right)^t \\
&= \sum_{j=1}^{t} z_{t-j+1} \left(\frac{1}{1+r}\right)^j + u_0 \left(\frac{1}{1+r}\right)^t \ . \qquad (33)
\end{aligned}
$$

Considering the last line of (33), the first term is a null-sequence as $z_t \to 0$. The second summand converges to zero as $t \to \infty$. This results in $\liminf_t u_t \geq 0$. Since, $\liminf_t u_t \geq 0$ and $\limsup_t u_t \leq 0$, the sequence (u_t) has to converge to zero. ■

Let us define $\tilde{p}_t := -p_t$ and $\tilde{z}_t := -z_t$; then $\sigma^2_{p,t} = \tilde{\sigma}^2_{p,t} = \sigma^2_{\tilde{p},t}$, $\beta^2_t = \tilde{\beta}^2_t$, and

$$
\begin{aligned}
\tilde{p}_t &= \frac{1}{1+r}\left(\tilde{z}_t + c\tilde{\sigma}^2_{p,t} + (1-\tilde{\beta}^2_t)\tilde{\alpha}_{p,t} + \tilde{\beta}^2_t\tilde{p}_{t-1}\right) \\
&= \frac{1}{1+r}\left(\tilde{z}_t + c\sigma^2_{p,t} + (1-\beta^2_t)\tilde{\alpha}_{p,t} + \beta^2_t\tilde{p}_{t-1}\right) \ . \qquad (34)
\end{aligned}
$$

Lemma 3 *For $0 < \nu < \frac{r}{c}$, $0 < \mu \le r\nu - c\nu^2$, $|p_0| \le \nu$, and $\sup_t |\tilde{z}_t| \le \mu$, then $|\tilde{p}_t| \le \nu$ for all t.*

Proof From equation (34), and the definition of the variance of prices (18) we derive:

$$
\begin{aligned}
\tilde{p}_t &= \frac{1}{1+r}\left(\tilde{z}_t + c\sigma^2_{p,t} + (1-\beta^2_t)\tilde{\alpha}_{p,t} + \beta^2_t\tilde{p}_{t-1}\right) \\
&\le \frac{1}{1+r}\left(\tilde{z}_t + \frac{c}{t}\sum_{i=0}^{t-1}\tilde{p}^2_i + (1-\beta^2_t)|\tilde{\alpha}_{p,t}| + \beta^2_t|\tilde{p}_{t-1}|\right) \ . \qquad (35)
\end{aligned}
$$

The proof of this lemma follows from induction. For $t = 1$ we have:

$$
\begin{aligned}
\tilde{p}_1 &\le \frac{1}{1+r}\left(\mu + cp^2_0 + (1-\beta^2_t)|p_0| + \beta^2_t|p_0|\right) \\
&\le \frac{1}{1+r}(r\nu + \nu) \ = \ \nu \ . \qquad (36)
\end{aligned}
$$

Let $|\tilde{p}_i| \le \nu$ for all $i < t$, then $|\alpha_{p,i}| \vee |\tilde{p}_i| \le \nu$. This yields:

$$
\begin{aligned}
\tilde{p}_t &\le \frac{1}{1+r}\left(\mu + c\nu^2 + |\alpha_{p,i}| \vee |\tilde{p}_{t-1}|\right) \\
&\le \frac{1}{1+r}(r\nu + \nu) \ = \ \nu \ , \qquad (37)
\end{aligned}
$$

which proves that $|\tilde{p}_t| \le \nu$ for all t. ■

Lemma 4 *Suppose that $(\tilde{p}_t)$ solves equation (34), and the requirements of Lemma 3 are satisfied, then (v_t) solves:*

$$
v_t = \frac{1}{1+r}\left(|\tilde{z}_t| + (1+c\nu)\{\alpha_{v,t} \vee v_{t-1}\}\right) \ , \qquad (38)
$$

where $\alpha_{v,t} := \frac{1}{t}\sum_{i=0}^{t-1} v_i$, and $v_0 = |\tilde{p}_0|$. Then $|\tilde{p}_t| \le v_t$ for all t.

Proof The Lemma holds for $t = 0$. Assume the Lemma is true for all $i < t$. Then as $\frac{1}{t}\sum_{i=0}^{t-1}\tilde{p}^2_i \le \nu\alpha_{v,t}$:

$$
\begin{aligned}
|\tilde{p}_t| &\le \frac{1}{1+r}\left(|\tilde{z}_t| + c\nu\alpha_{v,t} + (1-\beta^2_t)\alpha_{v,t} + \beta^2_t|\tilde{p}_{t-1}|\right) \\
&\le \frac{1}{1+r}\left(|\tilde{z}_t| + (1+c\nu)(\alpha_{v,t} \vee v_t)\right) \ = \ v_t \ . \qquad (39)
\end{aligned}
$$

■

Now we have obtained the necessary tools, to treat the problem of convergence analytically:

Theorem 1 *If* $0 < \nu < \frac{r}{c}$, $0 < \mu \leq r\nu - c\nu^2$, $|p_0| < \nu$, *and* $\sup_t |z_t| \leq \mu$, *then* $\lim_t p_t = 0$.

Proof From Lemma 1 we already know that $\limsup_t p_t \leq 0$. What remains to show is $\liminf_t p_t = -\limsup_t -p_t \geq 0$. Therefore, let $\tilde{p}_t = -p_t$, and (v_t) solves equation (38). Since v_t dominates $\tilde{p}_t = -p_t$ by Lemma 4, it is sufficient to show that $\lim_t v_t = 0$. Since $v_t \geq 0$ we prove $\limsup_t v_t = 0$. From the assumption $c\nu < r$ it follows that:

$$\frac{1 + c\nu}{1 + r} = \frac{1}{1 + \hat{r}} \ , \tag{40}$$

where $0 < \hat{r} := \frac{r - c\nu}{1 + c\nu}$. Let $w_t := \frac{|\tilde{z}_t|}{1 + c\nu}$, then the sequence (v_t) is the solution of:

$$v_t = \frac{1}{1 + \hat{r}} \left(w_t + \alpha_{v,t} \vee v_{t-1} \right) \ , \tag{41}$$

where (v_t) converges to 0 by Lemma 2. Thus, $\liminf_t p_t = -\limsup_t \tilde{p}_t \geq -\limsup_t v_t = 0$. Therefore $(\tilde{p}_t) = (-p_t)$ as well as (p_t) have to converge to zero. ∎

Remark 10 *At the beginning of this section we have transformed the sequence* (z_t) *to a sequence converging to zero.* p_t *will converge to zero, if the conditions in Proposition 1 are satisfied. However, if* $z_t \to z$ *with probability one, the steady-sate of our system (20) is given by* $p^* = \frac{z}{r}$. *By our sufficient conditions for convergence in Proposition 1,* $p_t \to p^*$ *and* $\theta_t \to \theta$ *(a.s.), if*

$$|p_0 - \frac{z}{r}| \leq \nu \leq \frac{r}{c} \ , \tag{42}$$

and

$$\sup_t |z_t - z| \leq r\nu - c\nu^2 \ . \tag{43}$$

However, the reader should note that these conditions are only sufficient for almost sure convergence.

B Bounded Support

First we derive the largest distance in inequality (24) as a function. This is done by deriving the minimum and the maximum of z_t. Therefore, let us consider a real valued sequence (x_i) on bounded support, i.e. $x_i \in [a, b]$. To obtain the infimum of

$$I^c(a, b) := \lim_{n \to \infty} \inf_{x_i} \left(\frac{\sum_{i=1}^n x_i}{n} - c \frac{\sum_{i=1}^n (x_i - \bar{x}_n)^2}{n} \mid x_i \in [a, b] \right) \ , \tag{44}$$

where $c > 0$, we transform x_i to y_i such that $y_i \in [0, 1]$, i.e. $x_i = a + (b - a)y_i$, and $I_n^c(a, b) = a + (b - a)I_n^{c(b-a)}(0, 1)$. The reader can easily check that the y_i minimizing the expression

$$\bar{y} - c\frac{\sum_{i=1}^{n} y_i^2}{n} - c\bar{y}^2$$
$$such\ that : y_i \in [0,1]\ and\ \bar{y} = \frac{\sum_i y_i}{n}\ , \tag{45}$$

is given by $y_i = \bar{y}$. $\bar{y}$ is derived from the constrained problem (45), yielding

$$\bar{y} = \frac{c-1}{2c}\ . \tag{46}$$

Inserting equation (46) into the first line of equation (45), and performing the transformation $I_n^c(a,b) = a + (b-a)I_n^{c(b-a)}(0,1)$ yields

$$I^c(a,b) = \{ \begin{array}{ll} a - \frac{(c(b-a)-1)^2}{4c} & for \quad c(b-a) > 1 \\ a & for \quad c(b-a) \leq 1 \end{array}\ . \tag{47}$$

On the other side, the supremum

$$S^c(a,b) := \lim_{n\to\infty} \sup_{x_i} \left(\frac{\sum_{i=1}^{n} x_i}{n} - c\frac{\sum_{i=1}^{n}(x_i - \bar{x}_n)^2}{n} \mid x_i \in [a,b] \right)\ , \tag{48}$$

is given by

$$S^c(a,b) = b\ . \tag{49}$$

Proof of Corollary 3:

Proof Let $x_t = d_t$, $a = d_l$, $b = d_h$, and suppose $c(d_h - d_l) \leq 1$. The length of the interval λ, is defined by $\lambda := d_h - d_l$. From the above considerations we already know that $|z - z_t| \leq |z - d_l| \vee |z - d_h|$, since $d_l \vee d_h$ are the extreme values of z_t, and $|z - d_l| \vee |z - d_h| := \max\{|z - d_l|, |z - d_h|\}$. From appendix A, convergence takes place if $|z - z_t| \leq \mu \leq r\nu - c\nu^2$, and $|p_0| \leq \nu \leq \frac{r}{c}$, where $\nu = \frac{r}{2c}$ minimizes the right side of the first expression. Summing up these results yields $0 \leq |z - d_l| \vee |z - d_h| \leq \mu \leq r\nu - c\nu^2$, where the ν minimizing $r\nu - c\nu^2$ and fulfilling $\nu \leq \frac{r}{c}$ is given by $\nu = \frac{r}{2c}$. This results in $|z-d_l| \vee |z-d_h| \leq \frac{r^2}{4c}$. Since, $z = \mu_d - c\sigma_d^2$, and $\mu_d - d_l > 0$, we derive $c \leq \frac{r^2}{4(\mu_d - d_l)} \leq \frac{r^2}{4(z-d_l)}$, which proves Corollary 3, and Remark 9. ■

Proof of Corollary 4:

Proof From relationship (47) and the conditions of Proposition 1 we obtain

$$c^2\lambda^2 + c\left(4z - 4d_l - 2\lambda\right) + 1 - r^2 \leq 0\ . \tag{50}$$

By setting (50) to zero, we derive the roots

$$c_{1,2} = -\frac{2(z-d_l)-\lambda}{\lambda^2} \pm \sqrt{\left(\frac{2(z-d_l)-\lambda}{\lambda^2}\right)^2 - \frac{1-r^2}{\lambda^2}}\ . \tag{51}$$

Since c has to be real valued, the term inside the square-root has to be nonnegative, i.e. the inequality $D := 4[(z - d_l)^2 - \lambda(z - d_l)] + r^2\lambda^2 \geq 0$ has to be satisfied. If this would be the case, then the solution of (51) is real valued. The assumptions that c is real valued and $c\lambda > 1$, require $D \geq 0$ and $c_2\lambda \geq 1$. Considering these requirements, and the c_2 derived in (51), yields:

$$r^2\lambda \geq 4(z - d_l) \ . \tag{52}$$

Additionally, the due to the second extreme value of z_t, the inequality $|z - d_h| \leq \frac{r^2}{4c}$, has to be satisfied as well, which proves Corollary 4. ■

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