Katrin Rabitsch and Serhiy Stepanchuk

A two-period model with portfolio choice: Understanding results from different solution methods

Article (Published)
(Refereed)

Original Citation:
ISSN 0165-1765

This version is available at: http://epub.wu.ac.at/4395/
Available in ePub^WU: December 2014

ePub^WU, the institutional repository of the WU Vienna University of Economics and Business, is provided by the University Library and the IT-Services. The aim is to enable open access to the scholarly output of the WU.

This document is the publisher-created published version. It is a verbatim copy of the publisher version.
A two-period model with portfolio choice: Understanding results from different solution methods

Katrin Rabitsch \(^a\), Serhiy Stepanchuk \(^c,1\)

\(^a\) University of Vienna, Department of Economics, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria
\(^b\) Vienna University of Economics and Business, Institute for International Economics and Development, Welthandelsplatz 1, A-1020 Vienna, Austria
\(^c\) École Polytechnique Fédérale de Lausanne, Chair of International Finance, Switzerland

**Highlights**
- We present a stylized two-period model of portfolio choice.
- We compare the solutions of Devereux-Sutherland and Judd-Guu with a nonlinear solution.
- The true portfolio solution depends on the size of uncertainty.
- The Judd-Guu method captures this dependence well.
- The Devereux-Sutherland solution is unaffected by changes in the size of uncertainty.

**Article Info**

*Article history:* Received 23 January 2014
Received in revised form 20 May 2014
Accepted 26 May 2014
Available online 2 June 2014

**JEL classification:**
E44
F41
G11
G15

**Keywords:**
Country portfolios
Solution methods
Size of uncertainty

**Abstract**

Using a stylized two-period model we compare portfolio solutions from two local solution approaches – the approach of Judd and Guu (2001) and the approach of Devereux and Sutherland (2010, 2011) – with the true nonlinear portfolio solution.

© 2014 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/3.0/).

1. Introduction

We present a stylized two-period model of portfolio choice and parameterize it to some key moments of returns on aggregate stock market indices. We use the model to compare the true nonlinear portfolio solution with the solutions from two approaches that belong to the class of local approximation methods, developed by Judd and Guu (2001, hereafter 'JG') and Devereux and Sutherland (2010, 2011, hereafter 'DS').

The DS solution approach has received considerable attention in solving portfolio problems in dynamic macroeconomic models in the recent past.\(^2\) While the two-period setting of the present paper ignores the main advantages of the DS method, which lie in obtaining portfolio solutions in dynamic settings (possibly in en-
vironments with many states variables), it nevertheless is able to shed light on some of its properties.\(^3\)

While DS and JG solution approaches are fundamentally similar, as they both are based on a Taylor-series approximation around the non-stochastic steady state, we find important differences between the results that they produce (as currently implemented). Devereux and Sutherland (2010, 2011) are mainly interested in incorporating the portfolio problem into dynamic macroeconomic models, and so they concentrate on approximating the solution in the direction of the model’s state variables, at the same time neglecting the effect of the size of the shocks.\(^4\) As a result, we find that in our two-period model, their approach delivers the constant portfolio solution independent of the size of the shocks.

At the same time, we show that the true solution generally depends on the size of uncertainty, with skewness, kurtosis and higher-order moments of the distribution of underlying shocks affecting the results. The JG bifurcation method is able to capture this dependency: its zero-order portfolio solution component affecting the results. The JG bifurcation method is able to capture the policy function for \(\sigma_e\), in a recursive economy, as a function that depends on the model’s state variables, \(x_t\), and on a parameter that scales the variance–covariance matrix of the model’s exogenous shock processes, \(\epsilon\); that is, \(\sigma_e = \alpha(x_t, \epsilon)\). In contrast to a standard Taylor series expansion to \(\sigma_e\), in a recursive economy, as a function that depends on the model’s state variables, \(x_t\), affect the optimal portfolio solution, but ignores the effect of variations in the size of uncertainty, \(\epsilon\). Because our model is static (we have \(\hat{\theta}_t = 0\)), the portfolio solution under DS is:

\[
\bar{\sigma}_e = \frac{\gamma_2 - \gamma_1}{\gamma_1(1 - \theta^0) + \gamma_2 \theta^0(1 - \theta^0)}.  \tag{2}
\]

Or, for \(\theta\):

\[
\theta = \theta^0 + \alpha \epsilon \rho \quad \text{where} \quad \alpha \epsilon = \bar{\sigma}_e. \tag{3}
\]

The property of \(\alpha_e\) which is key here, is that it is invariant to the size of the shock \(z\), and as a result, of any other statistical properties (skewness, kurtosis etc.).

To obtain the Judd–Guu portfolio approach, using bifurcation methods, we closely follow the steps outlined in Judd and Guu (2001). Unlike the DS approach, the JG solution depends on the size of uncertainty, and, as result, on higher-order moments of assets’ returns. Namely, the first-order terms of JG’s approximate solution depend on the returns’ skewness, while the second-order terms depend on their kurtosis.

2. Model

The world consists of two countries. In each there lives a representative investor for two periods, consuming a single consumption good in period 2. In period 1 investors decide on a portfolio over two assets: equity – a claim on the total world’s output – and a risk-free bond. The bond yields one unit of period-2-consumption and serves as numeraire, i.e., the period 1 bond price is normalized to 1. Each share has price \(p\) in period 1 and has a random period 2 value, \(Y = 1 + \epsilon z\). We assume \(E\{z\} = 0\) and \(E\{z^2\} = 1\). In addition, we assume that the support for \(z\) is bounded from below, so that \(Y > 0\) for all \(z\) and \(\epsilon\).

Each investor \(i\) starts with \(b^0_i\) units of bonds and \(\theta^0_i\) shares of equity. Investors’ utility is given by \(u_i(c_i) = C_i^{1-\gamma} / (1 - \gamma)\). \(C_i\) denotes investor \(i\)’s period-2-consumption which equals her final wealth. Without loss of generality, we assume \(\theta^0_i + \theta^0_i = 1\); this implies that \(z\) denotes aggregate risk in the world endowment \(Y\). Let \(b_i\) be the shares of equity and \(b_i\) bonds held by investor \(i\) after trading in period 1. Investor \(i\) solves:

\[
\max_{b_i, b_i} E[u_i(C_i)]
\]

s.t.:

\[
\theta^0_i\gamma p + b^0_i = \theta_i p + b_i \quad \text{(budget constraint in period 1)}
\]

\[
C_i = \theta_i Y + b_i, \quad \forall Y \quad \text{(budget constraints in period 2)}
\]

Market-clearing implies \(\theta_1 + \theta_2 = 1, b_1 + b_2 = 0\). Define \(\theta = \theta_1\); then \(\theta_2 = 1 - \theta\). Also, denote \(b_1 = b = -b_2\). Similarly, initial endowments \(\theta^0 = \theta^0_1, \theta^0_2 = 1 - \theta^0\), and \(b^0_i = b^0 = -b^0_2\). The model’s equilibrium conditions can be reduced to a system of two equations in \(\theta\) and \(p\):

\[
H(\theta, p, \epsilon) = \begin{bmatrix}
E[u(t(\theta Y + b^0 + (\theta^0 - \theta)p)(Y - p))]
\end{bmatrix} = 0. \tag{1}
\]

2.1. Portfolio solution methods

We comment only on the main points of the various portfolio solution approaches, and refer the interested reader to the online appendix for further documentation. To obtain the nonlinear (quadrature) portfolio solution in this simple economy, called ‘true solution’ hereafter, we approximate the expectations operator using quadrature methods and solve system (1) using a nonlinear equations solver.\(^6\)

To apply the Devereux and Sutherland approximation approach, which builds up on Samuelson (1970), we use DS’ notation convention and express portfolio holdings in terms of assets in zero-net supply, \(\alpha_x = [\alpha_x; \alpha_b] = [(\theta - \theta^0)p; b - b^0]\). Following Schmitt-Grohé and Uribe (2004) and Judd and Judd (2002) we can think of the true policy function for \(\alpha_x\), in a recursive economy, as a function that depends on the model’s state variables, \(x_t\), and on a parameter that scales the variance–covariance matrix of the model’s exogenous shock processes, \(\epsilon\); that is, \(\alpha_x = \alpha(x_t, \epsilon)\). In contrast to a standard Taylor series expansion to \(\alpha_x\), in a recursive economy, as a function that depends on the model’s state variables, \(x_t\), affect the optimal portfolio solution, but ignores the effect of variations in the size of uncertainty, \(\epsilon\). Because our model is static (we have \(\hat{\theta}_t = 0\)), the portfolio solution under DS is:

\[
\bar{\sigma}_e = \frac{\gamma_2 - \gamma_1}{\gamma_1(1 - \theta^0) + \gamma_2 \theta^0(1 - \theta^0)}. \tag{2}
\]

Or, for \(\theta\):

\[
\theta = \theta^0 + \alpha \epsilon \rho \quad \text{where} \quad \alpha \epsilon = \bar{\sigma}_e. \tag{3}
\]

The property of \(\alpha_e\) which is key here, is that it is invariant to the size of the shock \(z\), and as a result, of any other statistical properties (skewness, kurtosis etc.).

To obtain the Judd–Guu portfolio approach, using bifurcation methods, we closely follow the steps outlined in Judd and Guu (2001). Unlike the DS approach, the JG solution depends on the size of uncertainty, and, as result, on higher-order moments of assets’ returns. Namely, the first-order terms of JG’s approximate solution depend on the returns’ skewness, while the second-order terms depend on their kurtosis.

3. Results

Consider a setup of countries with identical initial endowments, \(b^0 = 0\) and \(\theta^0 = 0.5\) for country \(i = 1, 2\), but assume country 2 is twice as risk averse, reflected by \(\gamma_1 = \gamma_2 / 2\). In our numerical examples, we take the robust empirical stylized fact of positive and non-normally distributed equity premia seriously. We model world output endowment, \(Y = 1 + \epsilon z\), through a Normal-inverse

\(^3\) We perform a more extensive evaluation of the DS method, in a dynamic setting, in a companion paper (Rabitsch et al., 2014).

\(^4\) This is for simplicity of exposition. It also squares with the intuition that in standard macro models the size of the shocks does not affect the solution up to the first-order of approximation (see Schmitt-Grohé and Uribe (2004) and Judd and Judd (2002)). However, the JG solutions show that this intuition does not apply to a model with portfolio choice.

\(^5\) Since both approaches are based on Taylor series approximations, the intuition suggests that this should be possible. We thank the referee for this point.

\(^6\) In dynamic models global approximation methods would need to be applied, see e.g. Kübler and Schmedders (2003).
The N.I.G. distribution has experienced recent interest in the finance literature because of its flexibility in capturing the non-normal properties of asset pricing data. This gives us enough flexibility to target mean, standard deviation, skewness and kurtosis of equity (excess) returns in our model, to the observed moments of excess returns of aggregate stock market indices reported in Guidolin and Timmermann (2008), for Pacific-ex-Japan, United Kingdom, United States, Japan, Europe-ex-UK, and World, based on monthly MSCI indices—repeated in columns 1–4 of Table 1.\footnote{The N.I.G. distribution has experienced recent interest in the finance literature because of its flexibility in capturing the non-normal properties of asset pricing data (see e.g. (Colacito et al., 2012)).}

Fig. 1 plots the portfolio solution for country 1’s equity share, $\theta$, as a function of the size of uncertainty $\varepsilon$, for two illustrative examples: ‘United Kingdom’ (panel A) and ‘Pacific-ex-Japan’ (panel B). The first region's MSCI displays positive, the latter's negative skewness; both displays substantial kurtosis.

The solid red line displays the true portfolio solution: as country 1 is less risk averse, it chooses to hold a higher share of equity than initially endowed with ($\theta > \theta^*$ = 0.5), which it finances by going short in debt. Also, the solution for $\theta$ depends on the size of uncertainty: for the UK case we observe that country 1’s optimal share in equity initially increases, and then decreases, as $\varepsilon$ increases. For Pacific-ex-Japan $\theta$ continuously decreases.

The portfolio solution obtained by the Judd–Guu approach can help understand the mechanisms that drive these results in more detail. The positive skewness of the UK’s MSCI return index (0.75) leads to a positive slope of the first-order (linear) Judd–Guu solution: positive skewness means shifting more weight to ‘good’ outcomes, such that an investor would demand more of the risky asset. Positive skewness therefore works to increase country 1’s optimal equity holdings, $\theta$, as $\varepsilon$ increases. While this logic applies to both investors, JG show that the strength with which equity demand increases in such case depends on investors’ relative ‘skew-tolerance’. For the CRRA preference specification we use, skew-tolerance is always larger for the less risk-averse country, implying that country 1’s appetite for taking risk increases more strongly and its chosen equity position goes up under positive skewness as $\varepsilon$ increases.\footnote{In particular, for each MSCI index we consider, we choose 4 parameters of the N.I.G. distribution to make sure that $E[z^2]$ and $E[z^3]$ match the observed skewness and kurtosis of that MSCI index’ returns from the data, and that $E[z] = 0$ and $E[z^4] = 1$ (the normalization assumed by Judd and Guu (2001), which we follow here). Since $E[z^2] = 1$, we control the volatility of the return process through the choice of $\varepsilon$. In our model the variance of gross equity return, $R_{e}$, is given by $\text{var}(R_{e}) = \text{var} \left( \frac{1}{\varepsilon} \right) = \varepsilon^2 E[R_{e}]^2$, because $E[z^2] = 1$ and $E[E[R_{e}]] = 1/p$. Using this result, we set $\varepsilon = \sqrt{\frac{\text{std}(R_{e})}{|\text{mu}(R_{e})|}}$, where $\text{mu}(R_{e})$ is the net return in the data. Finally, we pick our final free parameter, $\gamma_2$, to match the observed mean excess equity return.}

Panel B, ‘Pacific-ex-Japan’, provides a different example: returns display negative skewness (−2.3). This implies that the return distribution is more heavily shifted towards ‘bad’ outcomes, so investors demand less of the risky asset. Since the skew-tolerance coefficient continues to be higher for country 1, but now, because of negative skewness, multiplies a negative number $E[z^3]$, the slope from the first-order part of the JG solution is negative: the less risk averse country 1 decreases its holdings of the risky assets as $\varepsilon$ increases. The second-order JG solution is able to capture the effects of kurtosis on the portfolio solution. MSCI return-indices of both regions are characterized by substantial kurtosis (10.3 for UK, 22.3 for Pacific-ex-Japan). Kurtosis means putting more weight to tail events, so as $\varepsilon$ increases, this leads an investor to reduce demand for the risky asset. Again, this logic applies to both investors, the relative strength of this effect depends on investors’ relative ‘kurtosis-tolerance’.

---

**Table 1**: Optimal equity holdings obtained by different portfolio solution methods; model calibrated to various regions’ return data on MSCI aggregate stock market indices by Guidolin and Timmermann (2008).

<table>
<thead>
<tr>
<th>Asset</th>
<th>Data</th>
<th>$\gamma_2/\gamma_1 = 2$</th>
<th>$\gamma_2/\gamma_1 = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean, %</td>
<td>SD, %</td>
<td>Skew</td>
</tr>
<tr>
<td>UK</td>
<td>0.7503</td>
<td>6.1898</td>
<td>0.7587</td>
</tr>
<tr>
<td>Pacific-ex-Japan</td>
<td>0.3892</td>
<td>7.0538</td>
<td>−2.2723</td>
</tr>
<tr>
<td>World</td>
<td>0.4560</td>
<td>5.174</td>
<td>−0.8711</td>
</tr>
<tr>
<td>US</td>
<td>0.5415</td>
<td>4.4825</td>
<td>−0.7084</td>
</tr>
<tr>
<td>Japan</td>
<td>0.3733</td>
<td>6.4830</td>
<td>0.0700</td>
</tr>
<tr>
<td>Europe-ex-UK</td>
<td>0.4158</td>
<td>5.0578</td>
<td>−0.5672</td>
</tr>
</tbody>
</table>
For CRRA preferences, kurtosis-tolerance is lower for the less risk-averse country, so that the reduction in the demand for the risky assets due to (excess) kurtosis is more pronounced for the less risk-averse country: as $\varepsilon$ increases, country 1’s equity share further decreases.

Finally, the black dashed line in Fig. 1 shows the results from applying the DS solution approach. The DS solution coincides with the constant (zero-order) component of the Judd–Guu solution. As explained in Section 2 the portfolio solution under DS is a function of state variables only, and not a direct function of the size of uncertainty, $\varepsilon$. Since, in this simple static model there is no variation in states, the obtained constant solution is not only the zero-order solution, but actually corresponds to the DS solution up to any order.

Table 1 reports the optimal portfolio solutions for all other regions, calibrated to the respective MSCI return indices. Columns 5–8 (9–12) report the true portfolio solutions, the (second-order) JG solution, and the DS solution, for the scenario in which country 2 is twice (three times) as risk averse as country 1. The largest discrepancies emerge for MSCI Pacific-ex-Japan: the difference to the true solution of the equity share obtained by the (second-order) JG solution is $-2.31\%$ ($-3.14\%$), the difference of the DS solution $-6.13\%$ ($-7.56\%$).

4. Conclusions

In a two-period model, calibrated to match the key moments of returns on aggregate stock market indices, we find that DS and JG solutions coincide in the limit where uncertainty vanishes, but else differ. As currently implemented, the DS approach does not account for the variations in the size of uncertainty (and its interactions with other statistical properties of returns, such as skewness and kurtosis), unlike JG. We show that the resulting discrepancy between the DS and JG solutions can be non-trivial. This makes extending the DS solution to take into account the effect of the size of uncertainty an interesting direction for future research.

References


---

\[ JG’s \text{ definition of ‘kurtosis-tolerance’ is given by } \kappa (C) = -\frac{1}{2} \left( \frac{\gamma + 1}{\gamma} \right) \frac{\kappa''(C)}{\kappa'(C)} \]

For CRRA preferences, $\kappa (C) = -\frac{1}{5} \frac{\gamma + 1}{\gamma} \frac{\kappa''(C)}{\kappa'(C)}$. Note that in this case $\frac{\kappa''(C)}{\kappa'(C)} > 0$. Therefore, with $\gamma_1 < \gamma_2$ we have $\kappa (C_1) < \kappa (C_2)$. 