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No-Arbitrage Bounds for Financial Scenarios

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Abstract

We derive no-arbitrage bounds for expected excess returns to generate scenarios used in financial applications. The bounds allow to distinguish three regions: one where arbitrage opportunities will never exist, a second where arbitrage may be present, and a third, where arbitrage opportunities will always exist. No-arbitrage bounds are derived in closed form for a given covariance matrix using the least possible number of scenarios. Empirical examples illustrate the practical potential of knowing these bounds.

Keywords: finance, scenarios, no-arbitrage bounds, financial optimization

1. Introduction

We are interested in constructing discrete scenarios and simulations of financial asset returns which are free from arbitrage. This requirement for financial optimization models has been pointed out by, among others, Klaassen (2002), Sodhi (2005), or Geyer et al. (2010, 2013). To that end, we investigate the theoretical relation between expected excess returns and the associated arbitrage opportunities. Our innovation is to provide bounds for expected excess returns to determine whether or not arbitrage is possible theoretically, before simulations have been carried out. These bounds hold for the least possible number of scenarios irrespective of the particular algorithm to be used, provided the algorithm matches the given covariance matrix.

The present paper can be put into the context of many methods\textsuperscript{1} which have been developed to obtain discrete approximations of continuous distributions. Such methods attempt to find a

\textsuperscript{1}Kall and Mayer (2011, p. 366) provide an overview; the most important methods are reviewed below.
compromise between (statistical) accuracy and the curse of dimensionality, or focus on computational efficiency. For example, in moment matching (see, e.g., Høyland and Wallace, 2001; Høyland et al., 2003) the pre-specified moments of asset returns are matched using a rather small number of scenarios (i.e. discrete mass points) to obtain statistically acceptable approximations of continuous distributions. As Klaassen (2002) has pointed out, the resulting scenarios (or trees) may allow for arbitrage opportunities, since this aspect is not controlled for in moment matching algorithms. Klaassen describes how arbitrage opportunities can be detected, and emphasizes the need to routinely check for arbitrage after each simulation. Alternatively, he suggests adding constraints to the nonlinear moment matching problem. While he points out that adding constraints “will complicate the numerical optimization” (p. 1516), he does not provide any (numerical) evidence on the severity of this complication, however. As a matter of fact, we are not aware of any moment matching or other scenario generation algorithm which includes such constraints to prevent arbitrage at the outset.

Klaassen (1998) considers the problem of reducing the size of an already arbitrage-free tree while maintaining the absence of arbitrage. He suggests aggregating trees across states and/or time to obtain smaller trees with comparable properties. However, since his approach only works under the risk-neutral measure, it is not suitable for financial applications like portfolio optimization as pointed out by Kouwenberg (2001) and Geyer et al. (2010, 2013).

Scenario reduction methods as proposed by, for example, Pflug (2001) or Heitsch and Römisch (2003, 2009) are driven by a similar objective. They retain only a few paths from a very large number of scenarios such that the approximate (i.e. “reduced”) and the original distribution are close in terms of some probability metric. However, applying scenario reduction techniques entails the risk of arriving at scenario trees which admit arbitrage opportunities. A necessary condition for the absence of arbitrage is that the branching factor (i.e., the number of arcs emanating from a node) at each node of the tree must at least equal the number of non-redundant assets in the optimization problem (e.g. Harrison and Kreps, 1979). While it is straightforward to use that minimum branching factor, resulting trees may still admit arbitrage opportunities, since that condition is not sufficient. This is complicated by the fact that existing implementations of scenario reduction algorithms do not allow for controlling the branching factor for each node in the tree. Pflug (2001) proposes a scenario reduction method to optimally discretize a continuous distribution
by minimizing the supremum of the distance between the objective function evaluated using the original probability distribution and its discrete approximation. However, Geyer et al. (2010) show that the equivalence of the original and approximated problems rests on the (implicit) assumption that the sup-distance is finite. This assumption is violated when there are arbitrage opportunities in the approximated problem. In principle, no-arbitrage constraints could be added to the discretization problem. However, deriving a constrained solution for that problem in closed form does not seem to be a trivial task. As it stands, trees obtained by scenario reduction must subsequently be checked for arbitrage.

King et al. (2005) use a Gauss-Hermite process for numerically computing bounds for the arbitrage-free prices of an option. An attractive feature of these processes is that the discretized one-step conditional probabilities match a maximum number of moments of the normal distribution (i.e. with ν branches, the Gauss-Hermite quadrature matches $2\nu - 1$ moments). The no-arbitrage bounds for an option are based on convex duality properties and require some other options available for trading. In this respect their paper also belongs to those which require solving an optimization problem to make statements about no-arbitrage (bounds).

This review of the literature makes clear that existing simulation methods mainly focus on statistical properties. The aspect of no-arbitrage, which is key in financial optimization, usually only enters ex-post. The (occasionally) proposed inclusion of constraints to guarantee no-arbitrage has not been implemented in any of the well-known scenario generation procedures we are aware of. In this paper we take a point of view which starts out from the no-arbitrage requirement. We investigate how the pre-specified first and second moments determine arbitrage possibilities in the discrete state space of simulated returns. This point of view and the associated results distinguish the present paper from the literature. Before simulations have been done, we are able to answer the question whether a vector of expected excess returns does allow for arbitrage or not. As a main contribution of the paper, this replaces the usual tests for arbitrage which require solving a linear program (see Klaassen, 2002). For a given covariance matrix we derive bounds for expected excess returns in closed form. Using well-known results from linear algebra and standard techniques from convex optimization we are able to distinguish three possible cases on the basis of two concentric (hyper)ellipsoids. These separate the space of all possible expected excess returns into three regions: (a) Arbitrage opportunities always exist. In this case simulation or re-sampling need not even be
attempted. For the application at hand the intended stochastic features of asset returns need to be reconsidered, i.e. assumptions about covariances and/or expected excess returns need to be altered, thereby taking into account that these parameters may have been estimated with large standard errors. (b) Arbitrage may or may not exist depending on the sample at hand. In this case arbitrage checks are still required to decide whether re-sampling is necessary or not. However, the distance of the vector of expected excess returns from the origin of the (hyper)ellipsoids indicates how likely the need for re-sampling is \textit{ex-ante}. This distance can also be used in cases (a) and (b) to quantify the required change in expected excess returns to guarantee no-arbitrage. (c) Arbitrage opportunities will never exist. Knowing this has the advantage that ex-post arbitrage checks become redundant.

To derive these bounds and to identify these regions we make use of the Fundamental Theorem of Asset Pricing. We only assume that the assets’ first and second moments exist, and make no assumptions about the distribution of returns. To account for the aspect of dimensionality we derive the bounds for the smallest possible state space. This implies that the associated trees have the least possible size but still match the first two moments of asset returns. We finally illustrate how information about bounds can be empirically derived and used.

In Section 2 we derive no-arbitrage bounds for expected excess returns. We also provide results from empirical illustrations in Section 3. Main features of the approach are summarized in Section 4.

2. No-arbitrage bounds

The results on no-arbitrage bounds derived below will hold \textit{irrespective} of a particular realization of excess returns $\mathbf{R}$. Thus, the no-arbitrage properties of expected excess returns $\mathbf{\mu}$ can be judged ex-ante (i.e. before simulations are run). However, for \textit{deriving} these bounds it is instructive to start by considering no-arbitrage conditions associated with a particular realization, and then generalize to any realization. In Section 2.1 we show how to simulate returns such that their covariance is exactly matched. In Section 2.2 we state necessary and sufficient conditions for no-arbitrage. In Section 2.3 we explore the discrete state space associated with simulated returns. We use the conditions established in Section 2.2 to derive general bounds for expected excess returns to rule out arbitrage opportunities in simulated returns (first for a particular realization and subsequently for any realization).
2.1. Return realizations

To simulate single-period returns we proceed in two steps. First, we generate realizations of *mean-zero* returns $Y$ with the target covariance $\Phi$. Second, we consider *excess* returns $R$ with mean $\mu$. All *numerical* results in the main part of the paper are based on normal random numbers. However, while any simulation requires distributional assumptions, we emphasize that all conclusions and closed-form results on no-arbitrage bounds derived in Section 2.3 *do not* depend on the distribution of returns. We only assume that first and second moments exist.

When defining the number of scenarios $m$ (i.e. the number of realizations of random returns) we need to account for three aspects: (a) According to Harrison and Kreps (1979) and Harrison and Pliska (1981) a necessary condition for no-arbitrage is $m \geq n$ (i.e. at least as many discrete states as assets). (b) As we are going to show in the next paragraph, more realizations than assets are required to exactly match the covariance matrix $\Phi$, which implies $m \geq n+1$. (c) Our objective is to match the covariance and to rule out arbitrage with the smallest possible scenarios (or trees) to safeguard against the curse of dimensionality. Therefore, for the rest of Section 2 we set $m \equiv n+1$; this leads to the minimal tree size, which satisfies the necessary condition for no-arbitrage and allows to match the covariance exactly. At the same time, this also corresponds to considering a complete market with $n$ risky assets and a risk-free asset.

To construct $Y$ we generate an $m \times n$ matrix of realizations $X$ which are required to have the following properties: each row is equally likely, the mean of each column is zero, the standard deviation of each column is one, and the columns of $X$ are orthogonal (i.e. $(X'X)/m = I$). In general, the columns of a simulated $m \times n$ matrix $\Xi$ of random numbers will only *roughly*\(^2\) have mean zero, standard deviation one, and $(\Xi'\Xi)/m \approx I$ (see Ledermann et al. (2011) for different approaches to achieve these goals). To achieve *numerically exact* properties we first standardize each column of $\Xi$, i.e. we subtract the column means from each element, and divide the differences by the standard deviation of these columns. For numerically exact orthogonality of $X$ we use the Cholesky decomposition to obtain the upper triangular matrix $C$ of the covariance of the standardized matrix $\Xi$, and compute $X = \Xi C^{-1}$. Since the Cholesky decomposition requires the covariance of $\Xi$ to be positive definite, we need more realizations than assets (i.e. $m \geq n+1$).

Mean zero returns $Y$ are defined on the basis of the Cholesky decomposition $H$ with the

\(^2\)This holds even for a very large of number of realizations.
property $\mathbf{H}'\mathbf{H}=\Phi$:

$$\mathbf{Y} = \mathbf{XH}.$$  

By adding expected excess returns $\mu$ to $\mathbf{Y}$ we obtain excess returns

$$\mathbf{R} = \mathbf{Y} + \mathbf{1}_m\mu'.$$  

(1)

2.2. No-arbitrage conditions

According to the Fundamental Theorem of Asset Pricing, no-arbitrage holds if there exists an equivalent risk-neutral measure such that expected excess returns are equal to zero (see Ingersoll, 1987; Pliska, 1997). Thus, we require a solution of

$$\begin{pmatrix} \mathbf{R}' \\ \mathbf{1}_m' \end{pmatrix} \mathbf{q} = \begin{pmatrix} \mathbf{0}_n \\ 1 \end{pmatrix},$$  

(2)

such that risk-neutral probabilities are strictly positive ($\mathbf{q}>0$). Rather than discarding and re-sampling $\mathbf{X}$ if (2) does not hold, we derive a range of admissible $\mu$ such that $\mathbf{q}>0$ and $\mathbf{1}_m'\mathbf{q}=1$.

In order to investigate no-arbitrage conditions for $\mathbf{R}$ we first show that the mean-zero returns $\mathbf{Y}$ are free of arbitrage, and consider the solution of

$$\begin{pmatrix} \mathbf{Y}' \\ \mathbf{1}_m' \end{pmatrix} \mathbf{q} = \begin{pmatrix} \mathbf{0}_n \\ 1 \end{pmatrix}.$$  

(3)

Subsequently, in Section 2.3, we are going to replace $\mathbf{0}_n$ by non-zero expected excess returns $\mu$ and derive bounds for $\mu$ such that (2) still holds.

For $m=n+1$ equation (3) has a unique solution iff $\begin{pmatrix} \mathbf{Y}' \\ \mathbf{1}_m' \end{pmatrix}$ has full rank $n+1$. To see why this property holds we consider both components of $\mathbf{Y}$. $\mathbf{X}$ is an orthogonal matrix with rank $n$. Since the matrix $\mathbf{H}$ has rank $n$, the product $\mathbf{Y}=\mathbf{XH}$ has rank $n$ as well. Since $\mathbf{Y}$ are excess returns of risky assets having non-zero variance we can rule out the case that one of the columns of $\mathbf{Y}$ has identical elements. Thus, the row vector of ones $\mathbf{1}_m'$ is not in the space spanned by the rows in the matrix $\mathbf{Y}$, and the rank of the concatenated matrix increases to $n+1$.

To determine the unique solution of (3), consider the first $n$ rows of the system. Since the columns of $\mathbf{Y}$ sum up to zero, any constant $c>0$ solves these equations. The only constant which also solves the last row is $q_i=1/m$. Hence, the constant vector $\mathbf{q}=(1/m)\mathbf{1}_m$ solves system (3), and by the argument above, this is the only solution. Thus, mean-zero returns $\mathbf{Y}$ are free of arbitrage.
2.3. Bounds for expected returns

2.3.1. Bounds for a particular realization

We now consider excess returns $R$ for which (2) must hold by the Fundamental Theorem of Asset Pricing. The final results derived in this subsection are valid for simultaneously changing all elements of $\mu$. However, we find it useful to start with a ceteris paribus approach. We first consider only $\mu_1$, the expected excess return of the first asset, and set all other elements of $\mu$ to zero. If $\mu_1$ differs from zero the corresponding solution

$$q = \left( Y' \right)^{-1} \begin{pmatrix} -\mu_1 \\ 0_{n-1} \end{pmatrix} = A \begin{pmatrix} -\mu_1 \\ 0_{n-1} \end{pmatrix}$$

(4)

allows us to derive ceteris paribus bounds for $\mu_1$ such that $0 < q < 1$. Thus, $q$ must be less than one, and it is sufficient to find an admissible range of $\mu_1$ such that $q > 0$. Let the $m \times m$ matrix $A$ with elements $a_{ij}$ denote the inverse in (4). Define the vector $\ell$ consisting of elements which solve $-\mu_1 a_{i1} + a_{im} = 0 \forall i$ for $\mu_1$. To make sure that $q > 0$, $\mu_1$ must be greater than the maximum of all negative elements and less than the minimum of all positive elements in $\ell$. The same conditions must hold for the $\mu_i$’s of other assets.

We illustrate the derivation of bounds by considering three assets using the covariance of case (L) in Table 1. Simulating $m = n + 1 = 4$ realizations we obtain

$$Y = \begin{pmatrix} 0.2685 & 0.4931 & 0.4444 \\ -0.2485 & -0.2621 & 0.3215 \\ 0.2308 & -0.2191 & -0.2315 \\ -0.2507 & -0.0120 & -0.5344 \end{pmatrix}$$

and

$$A = \left( \begin{pmatrix} Y' \\ 1_m' \end{pmatrix} \right)^{-1} = \begin{pmatrix} 0.2686 & 1.0886 & 0.3174 & 0.25 \\ -0.9412 & -0.8307 & 0.9281 & 0.25 \\ 1.7915 & -1.1702 & -0.3465 & 0.25 \\ -1.1189 & 0.9123 & -0.8989 & 0.25 \end{pmatrix}.$$
The vector \( \mathbf{q}=(0.25\ 0.25\ 0.25\ 0.25)' \) solves (3). We first consider the admissible range for \( \mu_1 \) such that (4) holds with \( 0<\mathbf{q}<\mathbf{1} \). The vector \( \ell \) is given by

\[
\ell = (0.9308 \quad -0.2656 \quad 0.1395 \quad -0.2234)',
\]

which implies that \(-0.2234<\mu_1<0.1395\). For example, setting \( \mu_1=0.1 \) (and retaining \( \mu_2=\mu_3=0 \)) we find that \( \mathbf{q}=(0.2231\ 0.3441\ 0.0709\ 0.3619)' \) solves (2) and the associated returns \( \mathbf{R} \) are free of arbitrage.

Accordingly, we find \(-0.2136<\mu_2<0.2296\), assuming \( \mu_1=\mu_3=0 \). The associated admissible area for \( \mu_1 \) and \( \mu_2 \), given \( \mu_3=0 \), is shown in Figure 1. For example, expected excess returns \( \mathbf{\mu}=(0.2\ 0.05\ 0.0)' \) are not compatible with no-arbitrage for the realizations in this example. In other words, using \( \mathbf{R} \) based on this realization \( \mathbf{Y} \) and this \( \mathbf{\mu} \) we cannot find a \( \mathbf{q}>0 \) which solves (2), while we do find a solution for \( \mathbf{\mu}=(0.05\ 0.05\ 0.0)' \), for example.

[Figure 1 about here.]

If we also allow \( \mu_3 \) to change, the resulting feasible region for \( \mathbf{\mu} \) which excludes arbitrage opportunities can be described by a polyhedron. Figure 2 shows two polyhedra which are associated with correlation matrices in Table 1. Both cases are based on the same realizations (which affect the specific shape of a polyhedron) to facilitate comparisons. Figure 2(L) also includes the plane for \( \mu_3=0 \), already shown in Figure 1. Note that the volume of the admissible polyhedron shrinks as correlations increase towards one. Another way of summarizing the volume of the polyhedron is to consider the eigenvalues\(^3\) of the covariance matrix. The more “dispersed” the standardized eigenvalues are, the smaller the size of the polyhedron. The role of eigenvalues will be expressed more explicitly below.

It is worth pointing out that the feasibility of \( \mathbf{\mu} \) cannot be inferred from the ceteris paribus bounds, but must be derived from the polyhedron. For example, in case (H) the bounds are given by \(-0.0588<\mu_1<0.0561\); \(-0.0439<\mu_2<0.0840\); \(-0.0538<\mu_3<0.0448\). Figure 2(b) shows expected excess returns \( \mathbf{\mu}=(0.06\ 0.08\ 0.1)' \), indicated by the bullet. This \( \mathbf{\mu} \) is feasible since it is located inside the polyhedron, while \( \mathbf{\mu}_x=(0.1\ 0.1\ 0.0)' \), indicated by \( \times \), is not feasible.

[Figure 2 about here.]

\(^3\)To simplify comparisons we consider the “standardized eigenvalues” which are derived from original eigenvalues by standardization such that they sum up to one.
2.3.2. General bounds

As pointed out above, the shape of the feasible polyhedron and the bounds for $\mu$ depend on the specific realization of $X$ (which translates into $Y$ and $R$). This raises the question to what extent random sampling variation may violate the no-arbitrage bounds derived in Section 2.3.1. For example, one may suspect that it is always possible to find a matrix $X$, consisting of extremely unfavorable realizations, which does allow for arbitrage (e.g. arguing on the basis of normal random variables, supported on the whole real line).

To deal with these concerns we first present evidence from simulated returns $Y$ for the three assets in Table 1, using $m=n+1=4$. By superimposing simulated polyhedra we find that the set of all possible polyhedra is characterized by two ellipsoids as indicated by Figure 3. The specific shape and size of the ellipsoids depend on the covariances. This is illustrated in Figure 4 by showing the ellipses corresponding to $\mu_3=0$ for the two cases in Table 1. These simulations show that it is not possible to choose arbitrarily extreme realizations even if the return distribution used in the simulation has infinite support (as in case of the normal distribution in this numerical example). More importantly, the ellipsoids obtained from simulations indicate that general no-arbitrage bounds can be derived analytically. We first focus on the inner ellipsoid. To know this ellipsoid and the associated inner bounds is highly relevant for the following reason: If $\mu$ is within that inner ellipsoid generated returns are guaranteed to be free of arbitrage!

In Section 2.3.1 we have described how no-arbitrage bounds depend on $A$, the inverse in (4). For the sequel it is important to note three features of $A$. First, the sum of the first $m-1$ columns is zero (the last column sums up to one). Second, the variance of the first $m-1$ columns is given by $\sigma^2=\text{diag}(\Phi^{-1})/m^2$. Third, these two properties hold irrespective of the realization $X$. To obtain the bounds associated with the inner ellipsoid we consider the maximum possible realization for a given column of $A$. Without loss of generality we maximize the first element $a_1$ in that column of $A$ subject to $\sum_i a_i=0$ and $\sum_i a_i^2=m\sigma^2$, where $\sigma^2$ is the variance of the column considered (i.e. the corresponding element of $\sigma^2$). This optimization problem can be simplified by noting that all remaining elements $a_i (i\neq1)$ must be identical if the goal is to maximize a single element $a_1$. 9
This can be derived formally from the Lagrangian of the problem, and intuitively by recognizing that any deviation of the remaining \( a_i \)'s from a constant must reduce the maximal \( a_1 \). Thus, the simplified optimization problem is given by

\[
\begin{align*}
\max a_1 \quad \text{s.t.} \\
a_1 + (m-1)a_c &= 0 \\
a_1^2 + (m-1)a_c^2 &= m\sigma^2.
\end{align*}
\] (5)

Solving this problem, we find that the maximum realization for a column of \( A \) is given by

\[a_1 = \pm \sqrt{(m-1)\sigma^2}, \quad a_c = -a_1/(m-1)\].

The corresponding inner bounds for the expected excess return associated with that column of \( A \) are given by

\[\mu^* = \pm (1/m)\sqrt{(m-1)\sigma^2} = \pm (\sqrt{(m-1)m\sigma})^{-1}\].

For example, in case (L) the first diagonal element of \( \Phi^{-1} \) is 21.6774 (irrespective of the simulated \( X \)). The maximum possible realization in the first column is \( a_1 = \pm 2.016 \) and the inner bounds are \( \mu^* = \pm 0.25/2.016 = \pm 0.124 \), which agrees with Figure 4(L) and Figure 5.

It remains to be shown that the inner bounds are indeed located on an ellipsoid. For that purpose we define the vector \( \mu^*_j = (\mu^* 0_{n-1})' \) and note that

\[(\mu^* 0_{n-1})' \Phi^{-1} (\mu^* 0_{n-1}) = (\mu^*)^2 m^2 \sigma^2 = (\sqrt{(m-1)m\sigma})^{-2} m^2 \sigma^2 = 1/(m-1)\].

The constant \( 1/(m-1) \) results for any bound at position \( j \) of a vector \( \mu^*_j \) (all remaining elements of \( \mu^*_j \) are zero). In other words, for the ceteris paribus bounds corresponding to column \( j \) of \( A \) we find \( \mu^*_j' \Phi^{-1} \mu^*_j = 1/(m-1) \) \( \forall j \).

An expression of the form \( x' \Phi^{-1} x = c \) defines an ellipsoid. In the present case we obtain \( c = 1/(m-1) \) for \( x = \mu^*_j \). Thus, the ceteris paribus bounds are located on an ellipsoid. Moreover, since these bounds rule out arbitrage, any vector of expected excess returns \( \mu \) must obey that same constraint to rule out arbitrage. More specifically, the bounds for any vector \( \mu \) (not necessarily a vector of ceteris paribus bounds) are characterized by the inner ellipsoid satisfying

\[\mu' \Phi^{-1} \mu = 1/(m-1)\]. \hfill (6)

The eigenvectors of \( \Phi^{-1} \) define the principal directions of the ellipsoid, and the reciprocals of the square roots of the eigenvalues of \( (m-1)\Phi^{-1} \) determine the lengths of the corresponding semi-axes (Meucci, 2007, p. 54).
We proceed in a similar way to derive the outer ellipsoid. The corresponding outer bounds $\mu^o$ are based on the solutions $a_c$ of problem (5); i.e. $\mu^o = \pm (1/m)/a_c$. For the example above, $a_c = -0.672$ and $\mu^o = \pm 0.25/0.672 = \pm 0.372$, consistent with Figures 4(L) and 5. We express $a_c$ in terms of $a_1$ to define $\mu^o = (m-1)\mu^*$. From

$$
(\mu^o \; 0_{n-1}) \Phi^{-1} (\mu^o \; 0_{n-1})' = ((m-1)\mu^*)^2 m^2 \sigma^2 = (m-1)
$$

we find that the constant on the right hand side of (6) for the outer ellipsoid is given by $m-1$.

What we have illustrated in Figures 3 and 4 are known as extremal volume ellipsoids in convex optimization. For given $\Phi$ the outer ellipsoid in Figure 5 (right panel) corresponds to the minimum volume L"owner-John ellipsoid for each possible polyhedron, and the inner ellipsoid is given by the maximum volume inscribed ellipsoid of each possible polyhedron (see Boyd and Vandenberghe, 2009, pp. 410). The left panel in Figure 5 is the corresponding cut through this ellipsoid for $\mu_3 = 0$. Finding the inner bounds is equivalent to maximizing the Mahalanobis distance of $\mu$ from the origin (for a fixed $\Phi$) while ensuring the absence of arbitrage (see Meucci, 2007, p. 51). Arbitrage is possible for a particular simulation if expected excess returns $\mu$ are located in region (b) between the dashed and the solid ellipse in Figure 5 (left panel) or between the inner and outer ellipsoid in Figure 5 (right panel). If $\mu$ is within the inner bounds in region (c), i.e. within the solid ellipse in Figure 5 (left panel) or the inner ellipsoid in Figure 5 (right panel), no-arbitrage is guaranteed in any simulated tree. Arbitrage will always exist in region (a), i.e. outside the dashed line in Figure 5 (left panel) or outside the outer ellipsoid (right panel).

Finally, we consider generating a symmetric distribution with mean zero by using antithetic simulation, i.e. by defining

$$
Z = \begin{pmatrix}
Y \\
-Y
\end{pmatrix}.
$$

In this case we can show that the polyhedron based on $Z$ becomes a symmetrical octahedron, having half the volume of the polyhedron associated with $Y$ (see Figure 6). The inner ellipsoid stays the same while the outer ellipsoid shrinks such that the Mahalanobis distance becomes one.\(^4\) We have thus generalized the results obtained above to symmetric distributions. Note that antithetic simulation implies that no-arbitrage bounds are now based on $2m$ (rather than $m$ in Section 2.3.2).

\(^4\)Results are available from the authors upon request.
Summarizing, we have derived bounds to distinguish three regions, characterizing the no-arbitrage properties associated with a given covariance matrix. While testing for arbitrage generally requires solving a linear program (see Klaassen, 2002), in cases (a) and (c) (see Figure 5), this check can be simply replaced by comparing the Mahalanobis distance to the theoretically derived bounds. These bounds are valid for any return distribution, provided its first and second moments exist.\(^5\)

3. Empirical examples

We use two different publicly available data sets to show the practical relevance of the bounds derived above. Our first example is based on the data set of Campbell et al. (2003).\(^6\) The authors provide quarterly data (from 1952 to 1999) for the Treasury bill rate, the excess stock return, excess bond return, dividend-price ratio, yield spread (between a five year zero-coupon bond and the bill rate) and inflation. Campbell et al. (2003) estimate a vector autoregressive (VAR) model with one lag to account for potential predictability in asset returns, and to obtain the necessary input parameters for solving a strategic asset allocation problem. We use these data to check whether scenario trees for stocks and bonds of minimal size fulfill the no-arbitrage condition in (2).\(^7\)

We use an expanding window approach, and estimate the VAR model based on the first 10 years of quarterly data (1952-1962). From the 6×6 covariance matrix of the residuals and the estimated VAR parameters we determine the 2×2 covariance matrix \(\Phi\) for stocks and bonds. We use the VAR parameters together with the most recent realization in order to calculate the expected excess returns \(\mu\) for the next quarter. The Mahalanobis distance is then given by \(\mu^T \Phi^{-1} \mu\). For two risky assets and \(m=3\), the inner bound is equal to 0.5 and the outer bound is equal to 2. We repeat estimating the model, and computing \(\mu\) as well as the distance each quarter for each additional quarterly observation of the vector time series.

\(^{5}\)Geyer et al. (2012) provide a geometric interpretation of the no-arbitrage regions derived in the present paper and analyze the general case \(m\geq n+1\). They show that the Mahalanobis distances for inner and outer bounds in such a setting are still \(1/(m-1)\) and \((m-1)\).

\(^{6}\)The data are available for free from http://scholar.harvard.edu/campbell/data.

\(^{7}\)Note that this check is not meant to detect arbitrage opportunities in market prices.
The resulting historical evolution of the Mahalanobis distance in Figure 7 shows that all three regions identified in Section 2.3 are reached. In 73.5% of the cases the distance is below the inner bound and no-arbitrage is guaranteed. 23.8% of the time the distance is between the inner and the outer bound, which requires solving an LP in order to check for arbitrage opportunities for the particular scenario at hand. In the remaining 2.7% of the cases any simulated scenario would imply an arbitrage opportunity. In such a situation there is no point in solving an asset allocation problem. It is clear that the stochastic features of asset returns have to be modified, either by changing the vector of expected excess returns, the covariance of returns, or both. One option is to proceed along the lines of the Black-Litterman (1992) approach, i.e. accounting for so-called views about expected excess returns. Since the Mahalanobis distance depends on the data frequency one may also consider another holding period (e.g. by changing the frequency or scaling mean and covariance accordingly). An advantage of using the Mahalanobis distance in the context of such modifications is that it quickly indicates whether the (intended) changes are sufficient to end up below the loose bound. It also indicates whether the usual LP-based arbitrage check is required or not.

The second example is based on the 10 Industry Portfolios prepared by Ken French. Each NYSE, AMEX, or NASDAQ stock is assigned to one of ten industry portfolios (Consumer Non-Durables, Consumer Durables, Manufacturing, Energy, Business Equipment, Telecom, Shops, Healthcare, Utilities and Others) according to its four-digit SIC code. Monthly returns for these ten sectors and one-month T-bill rates are provided for the period 1926-2012.

Other than in the first example expected excess returns and the covariance matrix are estimated using a ten-year rolling window. The resulting historical evolution of the Mahalanobis distance is shown in Figure 8. For ten risky assets and $m=11$, the inner bound is equal to 0.1 and the outer bound is equal to 10. Two out of the three regions identified in Section 2.3 are reached. In 51.4% of the cases the Mahalanobis distance is below the inner bound, i.e. no arbitrage check is needed. For the rest of the time the distance is between the inner and outer bound.

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8The data are available for free from [http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html](http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html).
Finally, we note that similar results can be found for other time series or other methods to estimate expected excess returns and the covariance matrix, i.e., as shown with the two examples above, the Mahalanobis distance is located in one of the three regions, although with varying relative frequencies.

4. Conclusion

We have derived no-arbitrage bounds for expected excess returns in the context of generating scenario trees or simulations used in financial applications. The bounds are derived in closed form for the least possible number of scenarios. They are valid for any simulation algorithm, provided it matches the given covariance matrix exactly. We are able to distinguish three regions: one where no-arbitrage holds in any simulation, a second where arbitrage may be present, and a third, where arbitrage opportunities will always exist. Knowing these bounds has the advantage that prior to running any simulations the properties of expected returns and covariances with respect to their arbitrage implications and the need for arbitrage checks can already be determined.

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References


Figure 1: Example of the feasible area for $\mu_1$ and $\mu_2$ of three correlated assets from case (L) in Table 1, assuming $\mu_3=0$. 
Figure 2: Examples of the feasible region for three correlated assets. (L) and (H) correspond to the two cases of low and high correlations in Table 1.
Figure 3: Feasible no-arbitrage polyhedra for $\mu_1$, $\mu_2$ and $\mu_3$ from 1000 different realizations for case (L) in Table 1. Polyhedra above the plane corresponding to $\mu_3=0$ are cut to illustrate the bounding ellipses for $\mu_1$ and $\mu_2$. 
Figure 4: Feasible no-arbitrage areas for $\mu_1$ and $\mu_2$ from 100 different realizations for case (L) and (H) in Table 1.
Figure 5: Characterization of no-arbitrage regions for case (L) in Table 1. The left panel holds for $\mu_3=0$. The right panel shows the no-arbitrage region for two different factor realizations which share the same inner and outer ellipsoids.
Figure 6: Characterization of no-arbitrage regions with antithetic variables (case (L) in Table 1). The figure shows the polyhedron for a specific realization $Y$, as well as the octahedron associated with the corresponding $Z$. The inner ellipsoid is the same for $Y$ and $Z$, while the outer ellipsoid shrinks such that the Mahalanobis distance becomes one.
Figure 7: Historical evolution of the Mahalanobis distance based on expected excess returns and covariances implied by expanding window estimates of the VAR model and using the data described in Campbell et al. (2003).
Figure 8: Historical evolution of the Mahalanobis distance for the 10 Industry Portfolios prepared by Ken French. Expected excess returns and the covariance matrix are estimated from a ten-year rolling window.
Table 1: Statistical properties of returns. The cases (L) and (H) refer to low and high correlations. “std. EV” refers to the standardized eigenvalues of the covariance matrix.

<table>
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<th>asset</th>
<th>sd</th>
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<th>std. EV</th>
<th>case (H) correlations</th>
<th>std. EV</th>
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<td>0.5 1.0 0.9 0.12</td>
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<tr>
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<td>0.40</td>
<td>0.3 0.4 1.0 0.12</td>
<td></td>
<td>0.8 0.9 1.0 0.01</td>
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