Controllability of Discrete–Time Two dimensional Single Input Recurrent Neural Networks

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CONTROLLABILITY OF DISCRETE–TIME TWO DIMENSIONAL SINGLE INPUT RECURRENT NEURAL NETWORKS

THOMAS STEINBERGER AND LUCAS ZINNER

ABSTRACT. This paper presents a complete characterization of controllability and reachability for the class of discrete–time recurrent neural networks with state space dimension 2 and single input.

1. Introduction

In this paper we deal with control systems in discrete time. In general, by an \( n \)-dimensional, \( m \)-input recurrent \( \sigma \)-net we mean a discrete time control system of the form

\[
x(t + 1) = \tilde{\sigma}^{(n)}(Ax(t) + Bu(t)),
\]

where \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \) and the map \( \sigma : \mathbb{R} \to \mathbb{R} \) denotes a sigmoid function. Here for each positive integer \( n \) we use \( \tilde{\sigma}^{(n)} \) to denote the diagonal mapping

\[
\tilde{\sigma}^{(n)} : \mathbb{R}^n \to \mathbb{R}^n : \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} \sigma(x_1) \\ \vdots \\ \sigma(x_n) \end{pmatrix}
\]

The spaces \( \mathbb{R}^n \) and \( \mathbb{R}^m \) are called respectively the state space and the input–value space of the net. We mention that linear systems studied in control theory are the \( \sigma \)-nets for which \( \sigma \) is the identity function.

In neural networks theory one interprets the equation (1) as representing the evolution of an ensemble of \( n \) "neurons", where each coordinate \( x_i \) of \( x \) is a real–valued variable which represent the internal state of the \( i \)th neuron, and each \( u_i, i = 1, \ldots, m \) of \( u \) is the external input signal. The coefficients \( A_{ij}, B_{ij} \) denote the weights or "synaptic strengths" of the various connections. Finally the transformation \( \sigma : \mathbb{R} \to \mathbb{R} \) is usually called the "activation function". One example of such functions is the hyperbolic tangent \( \sigma = \tanh \).

Questions of parameter identifiability and observability were studied for such models in [3] and [4]. Here we focus on problems of controllability. One of

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the first contribution in this area was a paper of F. Albertini and P. Dai Pra, cf. [2], where they dealt with the study of the forward accessibility. Recall that a system (1) is called forward accessible if from each initial state it is possible to reach by using appropriate inputs \( u(\cdot) \) an open set of the state space. They showed that forward accessibility holds provided that a certain independence property holds for \( \sigma \) and the matrix \( B \) belongs to a certain class \( B_{n,m} \) of matrices which were introduced in [1], see also [3].

In [7] (see also [5]) E. Sontag and H. Sussmann studied the question of controllability in the continuous-time setting. A system is called completely controllable if from each initial state it is possible to reach, by using appropriate inputs \( u(\cdot) \), the entire state space, not only some — maybe very "small" — open set. In the continuous-time setting the equation (1) may be read as

\[
\dot{x}(t) = \sigma^{(n)}(Ax(t) + Bu(t)),
\]

where \( A \) and \( B \) are as above and \( \sigma \) has to be at least locally Lipschitz. The main step in their proof is that they could show that the convex hull of the set \( \{\sigma^{(n)}(a + Bu), u \in \mathbb{R}^m\} \) contains an open (in \( \mathbb{R}^n \)) neighborhood of zero. This holds if \( B \in B_{n,m} \), \( A \) arbitrary and \( \sigma = \tanh \), but fails if \( \tanh \) is replaced by say arctan.

In the linear case, that is if \( \sigma \) denotes the identity function, the system is controllable if and only if \( \text{rank} R(A,B) = n \), where

\[
R(A,B) := [B,AB,A^2B,\ldots,A^{n-1}B].
\]

For a proof we refer the reader to [6].

In contrast to the above mentioned results we showed in [8] that in the discrete-time case the assumptions for complete controllability are quite restrictive, more precisely we showed that a system is completely controllable if and only if the matrix \( B \) has full rank \( n \). In this paper we focus on two dimensional recurrent nets with single input, that is

\[
\sigma^{(2)} : \mathbb{R}^2 \to \mathbb{R}^2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \sigma^{(2)}(A \begin{pmatrix} x \\ y \end{pmatrix} + Bu),
\]

where

\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}
\]

and \( u \in \mathbb{R} \). We call \( B \) admissible if \( b_i \neq 0 \) for \( i = 1,2 \). Note that this is more general than to assume \( B \in B_{2,1} \).

2. Notations and Basic Definitions

To begin with let us recall the basic definitions we use.

**Definition 2.1.** Let \( t = 0,1,2,\ldots \). A system \( \Sigma \) given by

\[
x(t + 1) = \sigma^{(n)}(Ax(t) + Bu(t))
\]

is called recurrent neural network (RNN) with initial state \( x_0 \) if \( x(0) = x_0 \), \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \) and \( \sigma : \mathbb{R} \to \mathbb{R} \) is a sigmoid function, that is \( \sigma \) is
odd, absolutely bounded by 1 and strictly increasing. Furthermore $\mathcal{X} = (-1, 1)^n$ denotes the state space, where $(-1, 1)^n$ is defined to be $n$ times of the cartesian product of the interval $(-1, 1)$.

We will focus on the case where $n = 2$ and $m = 1$ in the definition above and write $\sigma$ instead of $\sigma^{(2)}$. Let us note that all the usually used functions such as tanh and $\frac{2}{\pi}$arctan fulfill the conditions cited above. In fact it is sufficient that the function $\sigma$ is assumed to be bounded by some constant.

In the following we will use either Greek or bold letters to indicate elements of $\mathcal{X}$ or $\mathbb{R}^2$.

**Definition 2.2.** Let $\zeta, \eta$ be in $\mathcal{X}$. Then $\zeta$ can be reached from $\eta$ iff there exist $t \in \mathbb{N}$ such that $\mathbf{x}(0) = \eta$ and $\mathbf{x}(t) = \zeta$ with appropriate chosen $u(0), \ldots, u(t-1)$. One says $\zeta$ can be reached from $\eta$ (or $\eta$ can be controlled to $\zeta$) if this happens for at least one $t$.

We use the notation $\eta \leadsto^T \zeta$ to indicate that the state $\zeta$ can be reached from $\eta$. Sometimes we write $\eta \leadsto^T \zeta$ if $\zeta$ can be reached from $\eta$ in time $T$. We denote

$$C^t(\zeta) := \{ \eta \in \mathcal{X} : \eta \leadsto^s \zeta \text{ with } s \leq t \} \quad \text{and} \quad C(\zeta) = \bigcup_{t=0}^{\infty} C^t(\zeta)$$

and

$$R^t(\eta) := \{ \zeta \in \mathcal{X} : \eta \leadsto^s \zeta \text{ with } s \leq t \} \quad \text{and} \quad R(\eta) = \bigcup_{t=0}^{\infty} R^t(\eta).$$

Finally denote

$$R = \bigcup_{\eta \in \mathcal{X}} R(\eta).$$

**Remark 2.1.** It is very easy to see that $\eta \leadsto^T \zeta$ and $\zeta \leadsto^T \xi$ implies $\eta \leadsto^T \xi$. It is also clear that if $\eta \leadsto^{T_1} \zeta$ for some $T_1 > 0$ and if $0 < t < T_1$, then there is some $\xi \in \mathcal{X}$ such that $\eta \leadsto^{t} \xi$ and $\xi \leadsto^{T_1-t} \zeta$.

But it is not necessarily true that $\eta \leadsto^T \zeta$ implies $\zeta \leadsto^T \eta$.

**Definition 2.3.** A system $\Sigma$ is said to be completely controllable if for each $\eta, \zeta \in \mathcal{X}$ it holds that $\eta \leadsto^T \zeta$. It is said to be completely controllable in time $T$ if for each $\eta, \zeta \in \mathcal{X}$ it holds that $\eta \in C^T(\zeta)$ and vice versa.

In [8] the following is proved.

**Theorem 2.1.** Let $\Sigma$ be a RNN. Then $\Sigma$ is completely controllable if and only if the $n \times m$-matrix $B$ has full rank $n$ independent of the matrix $A$. Especially if $n = 2$ and $m = 1$ then $R \subseteq \mathcal{X}$.

On the other hand we will show in the next section that $\eta \leadsto^T 0$ for all $\eta \in \mathcal{X}$ provided $A$ and $B$ fulfill certain properties.
3. Statement and Proof of the Main Results

The following proposition shows that $0$ can be reached from arbitrary $x \in \mathcal{X}$ in time 2 if appropriate rank conditions are supposed.

**Proposition 3.1.** Let $\Sigma$ be a two dimensional single input recurrent neural network, let $B$ be admissible. If $\text{rank}[AB, B] = 2$ then $C^2(0) = \mathcal{X}$.

**Proof.** Let

$$x_0 = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$  

Then

$$x(1) = \begin{pmatrix} \sigma(a_{11}x + a_{12}y + b_1u(0)) \\ \sigma(a_{21}x + a_{22}y + b_2u(0)) \end{pmatrix}$$  

and

$$x(2) = \begin{pmatrix} \sigma(a_{11}\sigma(a_{11}x + a_{12}y + b_1u(0)) + a_{12}\sigma(a_{21}x + a_{22}y + b_2u(0)) + b_1u(1)) \\ \sigma(a_{21}\sigma(a_{11}x + a_{12}y + b_1u(0)) + a_{22}\sigma(a_{21}x + a_{22}y + b_2u(0)) + b_2u(1)) \end{pmatrix}.$$  

We have to show that $x(2) = 0$ for appropriate chosen $u(0), u(1)$.

**Case i:** Assume $b_1 \neq \pm b_2$. Then

$$\gamma : \mathbb{R} \to \mathcal{X}, u \mapsto \begin{pmatrix} \sigma(a_{11}x + a_{12}y + b_1u) \\ \sigma(a_{21}x + a_{22}y + b_2u) \end{pmatrix}$$

defines a curve which crosses each line through $0$ at least one time. If $x_0 \in \ker A$ there is nothing to show since we can choose $u(0) = 0$ to gain our result. Though let us assume $x_0 \notin \ker A$ and $0 \notin \gamma^*$, where $\gamma^*$ denotes the graph of $\gamma$ as a subset of $\mathcal{X}$. Since $\gamma$ crosses each line through $0$, for each vector $v \in \mathbb{R}^2$ we can find $u(0)$ such that $\gamma(u(0)) = \lambda \cdot v$ for some $\lambda$. Now choose $v$ such that $Av = B$ we find some $u(0)$ and get

$$x(2) = \begin{pmatrix} \sigma(\lambda b_1 + b_1u(1)) \\ \sigma(\lambda b_2 + b_2u(1)) \end{pmatrix}.$$  

Note that this can be done, since $B \notin \ker A$, otherwise $\text{rank}[AB, B] < 2$. Taking $u(1) = -\lambda$ yields to the desired result.

**Case ii:** Assume $b_1 = \pm b_2$. We prove the result for $b_1 = b_2$. If $b_1 = -b_2$ the proof is similar. Set $b_1 = b$. We write

$$x(1) = \begin{pmatrix} \sigma(a_{11}x + a_{12}y + bu(0)) \\ \sigma(a_{21}x + a_{22}y + bu(0)) \end{pmatrix}.$$  

If $a_{11}x + a_{12}y = a_{21}x + a_{22}y$ we may take $u(0) = -\frac{1}{b}(a_{21}x + a_{22}y)$ so that $x(1) = 0$ and we are done.

Though let us assume $a_{11}x + a_{12}y \neq a_{21}x + a_{22}y$. Then

$$\gamma : \mathbb{R} \to \mathcal{X}, u \mapsto \begin{pmatrix} \sigma(a_{11}x + a_{12}y + bu) \\ \sigma(a_{21}x + a_{22}y + bu) \end{pmatrix}.$$
gives a curve which crosses each line through 0 at least one time except the diagonal \( \Delta = \{ (v_1, v_2) \in \mathcal{X} : v_1 = v_2 \} \). Since \( B \not\in \ker A \) there exists a \( \mathbf{v} \) such that \( A\mathbf{v} = B \) and \( \mathbf{v} \not\in \Delta \). Again we find \( u(0) \) such that \( \gamma(u(0)) = \lambda \cdot \mathbf{v} \). Choosing \( u(1) = -\lambda \) gives \( \mathbf{x}(2) = 0 \) as desired. \( \square \)

**Example 3.1.** Let
\[
\mathbf{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]
If \( x_0 \neq y_0 \), then
\[
\mathbf{x}(t + 1) = \begin{pmatrix} \sigma(x(t) + u(t)) \\ \sigma(y(t) + u(t)) \end{pmatrix} \not\in \Delta \quad \text{for all} \quad t = 0, 1, \ldots.
\]
Though obviously \( C(0) = \Delta \subset \mathcal{X} \). In fact if \( x_0 > y_0 \) the region \( \mathcal{R}(\mathbf{x}_0) \) is contained in \( \{ \mathbf{x} = (x, y) \in \mathcal{X} : x > y \} \). Note that in this example the rank condition is not fulfilled, since \( \operatorname{rank}[AB, B] = 1 \).

**Example 3.2.** Let
\[
\mathbf{x}_0 = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]
such that \( \sigma(x - 2y) > 1/2 \). Then
\[
\mathbf{x}(1) = \begin{pmatrix} \sigma(x + u(0)) \\ \sigma(x - 2y) \end{pmatrix}
\]
and
\[
\mathbf{x}(2) = \begin{pmatrix} \sigma(\sigma(x + u(0)) + u(1)) \\ \sigma(\sigma(x + u(0)) - 2\sigma(x - 2y)) \end{pmatrix}.
\]
If \( \mathbf{x}(2) = 0 \) we would have \( \sigma(x + u(0)) = 2\sigma(x - 2y) > 1 \) by assumption on \( x \) and \( y \). But \( \sigma \) is bounded by 1 which yields a contradiction. Note that in this case the rank condition is fulfilled, since \( \operatorname{rank}[AB, B] = 2 \), but \( B \) is not admissible.

The next proposition shows that skipping the rank condition we can not hope to get controllability of \( 0 \) by increasing time.

**Proposition 3.2.** Let \( \Sigma \) be a two dimensional single input recurrent neural network, let \( B \) be admissible. If \( \mathbf{x} \not\sim 0 \) then \( \mathbf{x} \not\sim 0 \) for all \( t < \infty \).

**Proof.** Let \( \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \) and we suppose there exists some \( t > 2 \) such that
\[
\mathbf{x} \xrightarrow{t} 0, \quad \text{but} \quad \mathbf{x} \xrightarrow{t-1} 0.
\]
Since \( \mathbf{x} \not\sim 0 \) it follows by Proposition 3.1 that \( \operatorname{rank}[AB, B] \leq 1 \).

**Case i:** Assume \( AB = 0 \), that is \( B \in \ker A \).

Suppose there exists \( t \) such that \( \mathbf{x}(t + 1) = 0 \) we have \( A\mathbf{x}(t) = -Bu(t) \) where \( \mathbf{x}(t) \) is the state gained after \( t \) steps and \( u(t) \in \mathbb{R} \) is the appropriate chosen
input. It follows that \( B \in \text{image } A \) and hence \( B \in \text{image } A \cap \ker A \). This yields to a contradiction to the admissibility assumption on \( B \).

**Case ii:** Assume \( AB = \lambda B \) for some \( \lambda \in \mathbb{R} \).

If \( B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \) and \( b_1 \neq \pm b_2 \) we can follow the proof of Proposition 3.1 to get \( \zeta \nabla 0 \) for all \( \zeta \in \mathcal{X} \).

Therefore we assume \( b_1 = \pm b_2 \). Again we only prove the case \( b_1 = b_2 \) and write \( b_i = b \). Let

\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.
\]

Then \( \text{rank}[AB, B] \leq 1 \) implies \( a_{11} - a_{21} = a_{22} - a_{12} \). If \( a_{11} = a_{21} \) we may choose \( u(0) = -\frac{1}{b}(a_{11}x + a_{12}y) \) and we are done.

Suppose \( a_{11} \neq a_{21} \). By assumption there exists some \( t \) such that

\[
0 = x(t + 1) = \begin{pmatrix} x(t + 1) \\ y(t + 1) \end{pmatrix} = \begin{pmatrix} \sigma(a_{11}x(t) + a_{12}y(t) + bu(t)) \\ \sigma(a_{21}x(t) + a_{22}y(t) + b_2u(t)) \end{pmatrix}.
\]

This is equivalent to \( (a_{11} - a_{21})x(t) = (a_{22} - a_{12})y(t) \) and since \( a_{11} \neq a_{21} \) we get \( x(t) = y(t) \). Applying the same argument we conclude that \( x(t - 1) = y(t - 1) \). Though choosing \( u(t - 1) = -\frac{1}{b}(a_{11}x(t - 1) + a_{12}y(t - 1)) \) we get \( x(t) = 0 \), though \( x \nabla 0 \) and we are done.

The next indicates that some rank condition is necessary if one is interested in "large" regions of reachability.

**Proposition 3.3.** Let \( \Sigma \) be a two dimensional single input recurrent neural network, let \( B \) be admissible. If \( \text{rank}[A, B] < 2 \) then \( \mathcal{R} = \mathcal{R}(x) = \mathcal{R}^1(x) \) for all \( x \) and \( \mathcal{R}^1(x) \) equals \( \gamma^s \) where \( \gamma \) denotes the curve

\[
\gamma : \mathbb{R} \to \mathcal{X}, \quad u \mapsto \begin{pmatrix} \sigma(b_1u) \\ \sigma(b_2u) \end{pmatrix}.
\]

**Proof.** Let

\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.
\]

Since \( \text{rank}[A, B] < 2 \) it follows that there exist \( \lambda \) and \( \mu \) such that

\[
\lambda \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \mu \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.
\]

Hence for all \( \mathbf{v} \in \mathbb{R}^2 \) we find \( u \in \mathbb{R} \) such that \( Au = Bu \) which proves the proposition.

**Remark 3.1.** We note that the conclusion of Proposition 3.3 does not hold if \( \text{rank}[A, B] < 2 \) is replaced by \( \text{rank}[AB, B] < 2 \). Example 3.1 shows that \( \mathcal{R}(x) \neq \mathcal{R}(0) \) provided \( x \notin \Delta \). In fact \( \text{rank}[A, B] < 2 \) implies \( \text{rank}[AB, B] < 2 \), but the converse fails.
Lemma 3.1. Let \( \Sigma \) be a two dimensional single input recurrent neural network, let \( B \) be admissible. Let \( x, x_1, x_2 \in X \). We define the curve
\[
\gamma_x : \mathbb{R} \to X, \quad u \mapsto \sigma(Ax + Bu).
\]
If there exist \( u_1 \) and \( u_2 \) such that \( \gamma_{x_1}(u_1) = \gamma_{x_2}(u_2) \) then \( \gamma_{x_1}^* = \gamma_{x_2}^* \).

Proof. If \( \gamma_{x_1}(u_1) = \gamma_{x_2}(u_2) \) it follows that \( Ax_1 + Bu_1 = Ax_2 + Bu_2 \) and hence \( Ax_1 = Ax_2 + B(u_2 - u_1) \). Therefore
\[
\gamma_{x_1}(u) = \sigma(Ax_1 + Bu) = \sigma(Ax_2 + B(u_2 - u_1 + u)) = \gamma_{x_2}(u_2 - u_1 + u)
\]
which gives the desired result.

Proposition 3.4. Let \( \Sigma \) be a two dimensional single input recurrent neural network, let \( B \) be admissible. Let \( \operatorname{rank} A < 2 \) and \( \operatorname{rank}[A, B] = 2 \). Then for all \( x \in X \) we get \( \mathcal{R}(x) = \mathcal{R}^2(x) \), that means if any state can be reached from \( x \) in finite time it can be reached in at most two steps of iteration.

Proof. Let \( \zeta \in \mathcal{R}(x) \), then there exist \( t < \infty \) and \( u(0), \ldots, u(t) \) such that \( x(t + 1) = \zeta \) or equivalent \( \zeta \in \gamma^* \) where \( \gamma : u \mapsto \sigma(Ax(t) + Bu) \).

Claim 1: There exist \( u' \) and \( v' \) such that \( \gamma(u') = c_0(v') \) where
\[
c_0 : \mathbb{R} \to X, \quad v \mapsto \sigma(A\sigma(Ax_0 + Bu)).
\]
The curve \( c_0 \) may be interpreted as the curve of starting points with respect to \( x(2) \).

Suppose this claim is already proved we get
\[
\sigma(A\sigma(Ax_0 + Bu')) = \sigma(Ax(t) + Bu')
\]
or equivalent
\[
Ax(t) = A\sigma(Ax_0 + Bu') - Bu'
\]
Since \( \zeta \in \gamma^* \) there is an input \( u(t) \) such that \( \zeta = \sigma(Ax(t) + Bu(t)) \). Now choose new inputs \( \bar{u}(0) = v' \) and \( \bar{u}(1) = u(t) - u' \) we get
\[
\bar{x}(2) = \sigma(A\bar{x}(1) + B\bar{u}(1)) = \sigma(A\sigma(Ax_0 + B\bar{u}(0)) + B\bar{u}(1))
\]
\[
= \sigma(Ax(t) + Bu' + B\bar{u}(1)) = \sigma(Ax(t) + B\bar{u}(t)) = x(t + 1) = \zeta
\]
and the proposition follows.

To prove claim 1 we will first prove the following
Claim 2: \( \gamma \) crosses the curve \( c_{t-1} \) exactly one time, where
\[
c_{t-1} : \mathbb{R} \to X, \quad v \mapsto \sigma(A\sigma(Ax(t-1) + Bu)),
\]
that is, suppose there exist \( u_1, u_2 \) and \( v_1, v_2 \) such that \( \gamma(u_1) = c_{t-1}(v_1) \) and \( \gamma(u_2) = c_{t-1}(v_2) \) then \( u_1 = u_2 \) and \( v_1 = v_2 \).

It is clear by the definition that \( \gamma \) crosses the curve \( c_{t-1} \) since \( \gamma(0) = c_{t-1}(u(t-1)) \). Now suppose there exist \( u \neq 0 \) and \( v \neq u(t-1) \) such that \( \gamma(u) = c_{t-1}(v) \). Then easy calculation leads to
\[
A(\sigma(Ax(t-1) + Bu) - \sigma(Ax(t-1) + Bu(t-1))) = Bu
\]
and hence \( B \in \operatorname{image}A \) which contradicts \( \operatorname{rank}[A, B] = 2 \). Here we used the assumption that \( \operatorname{rank} A < 2 \). It remains to show that \( \gamma \) is not tangent
to $c_{L-1}$. But this follows by the observation that small perturbation of the input $u(t-1)$ would yield to a curve $\gamma$ which crosses $c_{L-1}$ at least two times which is not possible. Of course it is clear from Lemma 3.1 that there is at most one curve $\gamma$ which is tangent to $c_{L-1}$ and claim 2 is proved.
To prove claim 1 observe that
\[
\lim_{v \to +\infty} c_{L-1}(v) = \lim_{v \to +\infty} c_0(v)
\]
and therefore any curve crossing $c_{L-1}$ also crosses $c_0$, which proves the proposition.

\[\square\]

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