\(\alpha\)-Consistent Expectations Equilibria

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Abstract
We modify the concept of consistent expectations equilibria introduced in Hommes and Sorger (1998) in two ways: (i) the consistency condition requires that the probability that the agents reject their perceived law of motion in any period does not exceed a given level and (ii) there may exist exogenous stochastic shocks. The concept is illustrated by two examples using a linear economic system. In one of the examples consistency implies rational expectations, in the other example it does not.
1 Introduction

A characteristic feature of most economic systems is that expectations about future events influence the present state of an economic system. Therefore, the specification of how economic agents form their expectations is an important aspect of economic modeling. The dominant paradigm in economic theory is the rational expectations hypothesis which postulates that economic agents make the best possible use of available information. Recently, this hypothesis has been critically assessed by many authors, see e.g. Sargent (1903). As a result, several alternative assumptions about expectation formation have been proposed which make less demanding rationality assumptions. One example is the concept of consistent expectations which has been proposed by Hommes and Sorg (1993). It assumes that economic agents act as econometricians: they observe past realizations of relevant variables and use simple linear models to compute forecasts for future values of these variable. The consistency requirement that Hommes and Sorg (1993) impose is that the agents do not make systematic forecast errors in the sense that the empirical autocorrelation function of the observed data coincides with the theoretical autocorrelation function of the linear model that is used by the agents. Hommes and Sorg (1993) develop this concept in a framework without any exogenous shocks.

In the present note we extend the concept of consistent expectations equilibrium in two directions. First, we allow exogenous shocks and, second, we modify the consistency requirement. In the present modification, it is assumed that the agents perform a statistical test in each period to find out whether their model is correct or not. If the probability of rejection in any period is smaller than a given level, then we call the situation an $\alpha$-consistent expectations equilibrium ($\alpha$-CEE). The advantage of this consistency requirement is that it can be checked in every period. The consistency requirement used by Hommes and Sorg (1998), on the other hand, can only be checked at $t = 1$, because it involves the long-run average of the observations and the empirical autocorrelation coefficients, which are computed as limits of certain functions as time goes to infinity.

The paper is organized as follows. In section 2 we define the concept of $\alpha$-CEE and in section 3 we provide two examples of $\alpha$-CEE in a simple linear economic system. The first example shows that the unique rational expectations equilibrium of this system qualifies also as an $\alpha$-CEE, whereas the second example demonstrates that the rational expectations equilibrium is not the only $\alpha$-CEE of the economy.

2 Definition of an $\alpha$-CEE

Let us define an economy to be a pair $(F, \epsilon)$ where $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a function and $\epsilon = (\epsilon_t)_{t=0}^{\infty}$ is a sequence of independent and identically distributed (i.i.d.) random variables defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The stochastic process $\epsilon$ describes exogenous stochastic shocks. The (only) endogenous variable in the model will be referred to as the “price”. The actual price in period $t$, denoted by $p_t$, depends on the price which agents forecast for that period, $p^e_t$, and on the stochastic shock in that period, $\epsilon_t$. The relation between these variables is described by the function $F$, that is,

$$p_t = F(p^e_t, \epsilon_t). \quad (1)$$

Agents do neither know the function $F$ nor do they know that there is a functional relationship between $p_t$ and the forecast $p^e_t$. Instead, they believe that prices follow the AR(1) process

$$p_t = a + b p_{t-1} + \mu_t, \quad (2)$$

where $a$ and $b$ are real numbers and $(\mu_t)_{t=0}^{\infty}$ is an i.i.d. sequence of random variables with expectation 0 and (unknown) variance $\sigma^2_\mu \geq 0$. We call the pair $(a, b) \in \mathbb{R}^2$ the perceived law of motion (henceforth, PLM). The agents forecast the price in period $t$ by

$$p^e_t = a + b p_{t-1}, \quad (3)$$

because this minimizes the expected sum of squared prediction errors given the PLM (2).
For a given economy \((F, e)\) and PLM \((a, b)\), the implied actual law of motion (henceforth, ALM) is the pair \((F_{a,b}, e)\), where the function \(F_{a,b} : \mathbb{R}^2 \rightarrow \mathbb{R}\) is defined by \(F_{a,b}(p, e) = F(a + bp, e)\). This means that the actual prices follow the stochastic difference equation

\[ p_t = F_{a,b}(p_{t-1}, e_t) = F(a + bp_{t-1}, e_t). \]  

(4)

Except for the presence of stochastic shocks, the above model resembles that of Hommes and Sorger (1998). In that paper an equilibrium condition was formulated which requires that the autocorrelation function of the actual price sequence coincides with the autocorrelation function of the PLM (2). This is a consistency requirement which can only be checked at \(t = \infty\). In this paper we introduce an alternative consistency condition which requires that, in each period \(t \geq t_0\), the agents reject their PLM with probability smaller than a given level \(\alpha_t\).

The basic idea of our consistency condition is as follows. In each period \(t \geq t_0\) the agents perform a statistical test \(\phi_t\) in order to determine whether the price observations are consistent with their PLM.\(^1\) The significance level for the test applied in period \(t\) is denoted by \(\alpha_t\). Let \(\alpha = (\alpha_t)_{t=t_0}^{\infty}\) be a given sequence of such levels. We call a price process \((p_t)_{t=t_0}^{\infty}\) and a PLM \((a, b)\) an \(\alpha\)-consistent expectations equilibrium (\(\alpha\)-CEE) if the probability that the agents reject the PLM in period \(t\) is smaller than \(\alpha_t\). In the formal definition of an \(\alpha\)-CEE (Definition 1 below) we allow for a general sequence of tests \((\phi_t)_{t=t_0}^{\infty}\). For example, \(\phi_t\) could test the null hypothesis that prices are generated by the PLM \((a, b)\) against the alternative that prices are generated by the AR(1) process \(p_t = \alpha + \beta p_{t-1} + \mu_t\) with \(\beta \neq (a, b)\). In this case, the parameter space of the test would be \(\mathbb{R}^2\) and the null hypothesis would be the singleton \(\{(a, b)\}\). Alternatively, \(\phi_t\) could test whether the residuals \(\mu_t = p_t - a - bp_{t-1}\) are white noise. Such a test might have a larger parameter space; see section 3 below for an example. We assume that all tests \(\phi_t, t \geq t_0\), have a common parameter space, which we call \(\Theta\). Moreover, we denote by \(H_0\) the subset of the parameter space which corresponds to the null hypothesis (which is that equation (2) describes the true price dynamics).

Let \(P_\theta\) be the probability distribution of the stochastic process \((p_t(\omega))_{t=t_0}^{\infty}\) defined by the ALM (4) under the assumption that the true parameter is \(\theta \in \Theta\). In the first example of tests \((\phi_t)_{t=t_0}^{\infty}\) given above this means that, if \(\theta = (\alpha, \beta)\), then \(P_\theta\) is the probability distribution of the sequence \((p_t)_{t=t_0}^{\infty}\) defined by the stochastic differential equation \(p_t = F(\alpha + \beta p_{t-1}, e_t)\). Analogously, let \(P_\theta\) be the probability distribution of the finite sequence \((p_t(\omega))_{t=0}^{t}\) under the assumption that the true parameter is \(\theta\). The (non-randomized) test in period \(t\) is a measurable function \(\phi_t : \mathbb{R}^{t+1} \rightarrow \{0, 1\}\) with the following interpretation: if \(\phi_t\) takes the value 0 the null hypothesis is accepted, and if \(\phi_t\) takes the value 1 the null hypothesis is rejected. The sample space for the test \(\phi_t\) is the set of all possible price observations during periods \(0\) to \(t\), that is, the argument of \(\phi_t\) is the \((t+1)\)-dimensional vector \((p_t(\omega))_{t=0}^{t}\).

**Definition 1** Let \((F, e)\) be an economy, \(\alpha = (\alpha_t)_{t=t_0}^{\infty}\), a sequence of real numbers \(\alpha_t \in (0, 1)\), and \((\phi_t)_{t=t_0}^{\infty}\) a sequence of tests with common parameter space \(\Theta\). The pair \((p_t)_{t=t_0}^{\infty}, (a, b)\) consisting of a stochastic process \((p_t)_{t=t_0}^{\infty}\) and a PLM \((a, b)\) is an \(\alpha\)-consistent expectations equilibrium (\(\alpha\)-CEE) if the following conditions are true:

(i) there exists a subset \(H_0\) of \(\Theta\) which corresponds to the validity of the PLM \((a, b)\).

(ii) \((p_t)_{t=t_0}^{\infty}\) satisfies the ALM of \((F, e)\) under the PLM \((a, b)\), that is, equation (4) holds for all \(t\) and almost all \(\omega\).

(iii) the probability under the null hypothesis that the test \(\phi_t\) rejects the null hypothesis is smaller than \(\alpha_t\), that is, for all \(\theta \in H_0\) and all \(t \geq t_0\)

\[ P_\theta\{(p_t)_{t=t_0}^{\infty} \mid \phi_t(p_0, p_1, \ldots, p_t) = 1\} < \alpha_t. \]

If \(\sum_{t=t_0}^{\infty} \alpha_t < 1\), then it follows that the probability that the PLM is never rejected is positive. That is, there is a positive probability that the agents never detect that their PLM differs from the ALM. Of course, such a situation can only occur if the significance levels \(\alpha_t\) converge to 0 fast enough.

It is clear that the smaller the numbers \(\alpha_t\) are, or the faster they converge to 0, the more restrictive is the consistency requirement. For \(\alpha_t = 1\) for all \(t\) the concept of \(\alpha\)-CEE has no bite at all, but for \(\alpha_t\) chosen sufficiently close to 0 it can be quite useful.

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1It is assumed that, for periods \(t < t_0\), too few observations are available to perform a meaningful test.

2For these definitions and further explanations on hypothesis testing see, e.g., Strawser (1985).
3 Examples of $\alpha$-CEE

In this section we describe two simple examples of $\alpha$-CEE in a linear economy. In the first example we verify that the rational expectations equilibrium is an $\alpha$-CEE for the modified Box-Pierce $Q$ test, and in the second example we show that an $\alpha$-CEE in the linear economy can be different from the rational expectations equilibrium (for a different specification of the tests).

Let us assume that the economy is described by the linear function

$$F(p; \alpha, \beta) = A + Bp + \epsilon_t$$

with $B \neq 1$ and that the stochastic shocks $(\epsilon_t)_{t=0}^\infty$ are i.i.d. normally distributed with mean 0 and variance $\sigma^2$. It is easily verified that the unique rational expectations equilibrium is the stochastic process $(\mu_t)_{t=0}^\infty$ defined by

$$\mu_t = \frac{A}{1-B} + \epsilon_t.$$  \hspace{1cm} (5)

3.1 Rational expectations equilibrium as $\alpha$-CEE

We are now going to show that the process defined in (5) together with the PLM

$$(\alpha, \beta) = (A/(1-B), 0)$$  \hspace{1cm} (6)

can constitute an $\alpha$-CEE if the agents use the modified Box-Pierce $Q$ test [Box and Pierce (1970), Box and Ljung (1978)]. We start by verifying condition (ii) of Definition 1. Substituting the specification of the economy and the PLM into (4) yields

$$\mu_t = F_{\alpha, \beta}(\mu_{t-1}, \epsilon_t) = A + B(\alpha + bp_{t-1}) + \epsilon_t = A + AB/(1-B) + \epsilon_t = A/(1-B) + \epsilon_t.$$ 

This proves that the process $(\mu_t)_{t=0}^\infty$ satisfies the ALM. It remains to verify conditions (i) and (iii) of Definition 1. Choose significance levels $\alpha_t$ such that $\alpha_t \in (0,1)$ for all $t \geq t_0$. Now observe that testing for the validity of the PLM $(\alpha, \beta)$ is equivalent to testing that the residuals $\mu_t = p_t - a - bp_{t-1}$ are white noise. The modified $Q$ test does exactly that. It tests for goodness of fit of a given model (here an AR(1) model), the null hypothesis being that the residuals are white noise, and the alternative being indeterminate. Suppose that $m$ is an arbitrary but fixed integer such that $1 \leq m \leq t_0$. In period $t \geq t_0$ agents compute the test statistic

$$Q_{m,t} = T(T+2) \sum_{k=1}^{m} \frac{r_k^2}{T-k},$$

where $r_1, r_2, \ldots, r_m$ are the autocorrelations defined by

$$r_k = \frac{T}{T} \sum_{s=k+1}^{T} \mu_s \mu_{s-k}$$

and $T = t-1$ is the number of residuals observed so far. If the null hypothesis is true (that is, if the residuals form a white noise process) then $Q_{m,t}$ is $\chi^2$ distributed with $m-1$ degrees of freedom. This fact allows one to construct an appropriate test for the given significance level $\alpha_t$. Finally note that, because of (5) and (6), it follows that the residuals satisfy

$$\mu_t = p_t - a - bp_{t-1} = p_t - A/(1-B) = \epsilon_t.$$ 

Thus, under the given PLM, the residuals are indeed white noise and the probability that the agents in period $t$ reject their PLM is smaller than $\alpha_t$. This proves that the rational expectations equilibrium together with the PLM stated in (6) qualifies as an $\alpha$-CEE.
3.2 An $\alpha$-CEE different from the rational expectations equilibrium

Consider the special case of the linear economy in which $A = 0$ and assume that the agents have the PLM $(a, b) = (0, b)$. The unique rational expectations equilibrium is therefore determined by $p_t = \epsilon_t$. The ALM, on the other hand, is given by

$$p_t = B b p_{t-1} + \epsilon_t.$$

The testing procedure $\phi_t$ used by agents is the following:

a) with the data available in period $t$, the agents compute the ordinary least squares estimator of the parameter $b$ in their PLM, that is, they compute

$$\hat{b}_t = \frac{\sum_{s=1}^{t} p_{s-1} p_s}{\sum_{s=1}^{t} p_{s-1}^2}.$$

b) in period $t$, agents accept the PLM if $\hat{b}_t$ lies inside a given interval centered around $b$. More precisely, the test in period $t$ is given by

$$\phi_t(p_0, p_1, \ldots, p_t) = \begin{cases} 0 & \text{if } \hat{b}_t \in (b - \gamma_t, b + \gamma_t), \\ 1 & \text{otherwise}, \end{cases}$$

where $\gamma_t$ is positive for all $t$. Since $\epsilon_t$ is independent from $p_0, p_1, \ldots, p_{t-1}$ it follows that $\hat{b}_t$ is asymptotically normally distributed with mean $B b$ and variance $\sigma^2/t$. Given the testing procedure specified above, and assuming $t_0$ large enough, the probability of rejecting the null hypothesis in period $t \geq t_0$ is approximately equal to

$$\bar{\alpha}_t = 1 - \int_{b - \gamma_t}^{b + \gamma_t} \frac{\sqrt{t}}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{t}{2} \left( \frac{x - B b}{\sigma} \right)^2 \right\} dx.$$

It is clear that $\bar{\alpha}_t \in [0, 1]$ independently of the choice of the values $\gamma_t$. For the case of a constant sequence, $\gamma_t = \bar{\gamma} > 0$ for all $t$, we have

$$\lim_{t \to \infty} \bar{\alpha}_t = \begin{cases} 1 & \text{if } B b \notin [b - \bar{\gamma}, b + \bar{\gamma}], \\ 1/2 & \text{if } B b \in \{b - \bar{\gamma}, b + \bar{\gamma}\}, \\ 0 & \text{if } B b \in (b - \bar{\gamma}, b + \bar{\gamma}). \end{cases}$$

We see that even if the PLM is different from the rational expectations equilibrium, that is, if $b \neq 0$, the PLM $(0, b)$ can qualify as an $\alpha$-CEE. If one would choose a sufficiently fast increasing sequence $\gamma_t$, then it would also be possible to obtain $\sum_{t=t_0}^{\infty} \bar{\alpha}_t < 1$ so that there is positive probability that the agents can never reject their PLM.
References


