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Report No. 43
December 1999
December 1999

SFB
‘Adaptive Information Systems and Modelling in Economics and Management Science’

Vienna University of Economics and Business Administration
Augasse 2–6, 1090 Wien, Austria

in cooperation with
University of Vienna
Vienna University of Technology

http://www.wu-wien.ac.at/am

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This piece of research was supported by the Austrian Science Foundation (FWF) under grant SFB#010 (‘Adaptive Information Systems and Modelling in Economics and Management Science’).
ON QUANTITATIVE APPROXIMATION OF STOCHASTIC INTEGRALS WITH RESPECT TO THE GEOMETRIC BROWNIAN MOTION

STEFAN GEISS

Abstract. For the geometric Brownian motion \( S_t = \exp(W_t - t/2), \ t \in [0, T] \), and a Borel-measurable function \( f : [0, \infty) \to [0, \infty] \) such that \( f(S_T) \in L_2 \), we approximate \( f(S_T) - \mathbb{E}[f(S_T)] \) in \( L_2 \) by expressions of type \( \sum_{i=1}^{n} v_{i-1} (S_{t_i} - S_{t_{i-1}}) \), where \( 0 = t_0 < \cdots < t_n = T \) are deterministic but not necessarily equidistant, the \( v_i \) are \( \mathcal{F}_{t_i} \)-measurable, and \( (\mathcal{F}_t)_{t \in [0,T]} \) is generated by \((W_t)_{t \in [0,T]}\). We ask for rates of convergence as \( n \to \infty \) and give information about optimal time knots \((t_i)_{i=1}^n\). Moreover, in the case of the standard European-Call and Put Option we show that the convergence rates with respect to \( L_2 \) and \( \text{BMO}^2 \) are the same.

1. Introduction

The construction of stochastic integrals consists usually of two steps. Firstly one defines stochastic integrals over simple integrands, secondly one extends this definition to more general integrands by a suitable approximation. For this construction the approximation rates occurring in the second step are not of interest in general. However, besides from the purely theoretical interest, there are situations, in which the knowledge of such approximation rates is useful.

Let us consider a typical situation in financial mathematics, where we assume for simplicity that the risk-less interest rate is equal to zero. Assume a semi-martingale \( S = (S_t)_{t \in [0,T]} \), where \( S_t > 0 \) stands for the price of a risky asset at time \( t \), and a random variable \( f(S_T) \) (the function \( f : [0, \infty) \to [0, \infty] \) is being Borel-measurable) describing the pay-off of an European option. Assume that

\[
    f(S_T) = u_0 + \int_0^T H_u dS_u,
\]

so that \( H_t \) can be seen as the position in the risky asset at time \( t \) in a continuously adjusted and self-financing portfolio, which hedges the pay-off \( f(S_T) \). In practice one has to replace the continuously adjusted hedging portfolio by a discretely adjusted one. Keeping the initial value \( u_0 \) and trading at time knots \( \tau = (t_i)_{i=1}^n \) this yields to the hedging error

\[
    \text{Err}(\tau) := \int_0^T H_u dS_u - \sum_{i=1}^{n} v_{i-1} (S_{t_i} - S_{t_{i-1}}),
\]

where the \( \mathcal{F}_{t_{i-1}} \)-measurable \( v_{i-1} \) describe the positions in the discretely adjusted portfolio. If one wishes to trade \( n \) times only, then it is naturally to ask for

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**Key words and phrases.** Geometric Brownian motion, stochastic integral, quantitative approximation, Kunita-Watanabe decomposition, weighted \( \text{BMO} \).

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0 = t_0 < \cdots < t_n = T$ such that the error $\text{Err}(\tau)$ becomes minimal in some sense. In particular, the knowledge of this error is of interest in the presence of transaction-costs if one wants to balance the hedging-costs and the expected error in a certain sense.

There are recent papers dealing with this question. For example Zhang [12] and Gobet and Temam [4] discuss the approximation using equidistant deterministic time nets and give corresponding approximation rates. On the other hand Martini and Patry [7] describe the existence and uniqueness of strategies using random nets with a fixed number of trading dates (without approximation rates). Our paper deals with the following questions:

(i) As pointed out by several authors it is not natural to use equidistant nets in all cases, so that random nets are usually proposed. However in between equidistant nets and random nets there is an appropriate alternative: Deterministic nets which are not necessarily equidistant. We consider such nets and show that there are cases in which one can improve the approximation order considerable: For example, the $L_2$-approximation order of the Binary Option with equidistant nets is $1/\sqrt{n}$, as obtained in [4]. Using special non-equitidistant nets one can realize an approximation order $1/\sqrt{n}$ as shown by Corollary 4.13 and Theorem 5.1 of this paper.

(ii) We give information about optimal non-equidistant nets by estimating the quantity $\overline{D}(f(S_T))$ from Definition 4.9 in Theorems 4.10 and 4.11. The quantity $\overline{D}(f(S_T))$ is related to the 'density' in the time interval $[t_0, T]$ of the knots belonging to these optimal nets as the cardinality of the nets tends to infinity.

(iii) In this paper the stochastic integrals are approximated in two different ways: A simple approximation (measured by the quantities from Definition 4.1 having the super-script "sim") and a best approximation (the corresponding quantities have the super-script "KW"). We show in Theorem 4.3 and Theorem 4.4, that the simple approximation is asymptotically as good as the best approximation. In the case of the approximation of $f(S_T)$, where $f$ is absolutely continuous and satisfies certain growth-conditions, this was shown in [12]. However the results of [12] do not apply, for example, to $f(S_T) = \mathbf{1}_{[K, \infty)}(S_T)$ the pay-off of the Binary Option, which is included in this paper (also note that [12](Théorème 2.1.2) does not hold true for the Binary Option).

(iv) To prove the above results we give in Theorem 3.1 a closed form expression for the approximation error for one step in a quite general setting.

(v) In most cases the approximation error is discussed in $L_2$. In the case of the European–Call and Put Options we measure the error much more restrictive by an appropriate $\text{BM}O$-quantity. This $\text{BM}O$-quantity fits completely with the consideration of the so-called risk process.

The paper is organized in the following way: After stating the setting and some preliminaries we consider in Section 3 the one-step approximation. We continue with the multi-step approximation in Section 4. As concrete examples we shortly treat the European–Call Option and the Binary Option in Section 5. Finally, in Section 6 we consider the approximation error of the European–Call and Put Option with respect to $\text{BM}O^2$. 
Although the results of this paper were obtained in the framework of the SFB 10 at the Technical University Vienna independently from [4], the paper should be seen as a continuation of [4]. Before getting knowledge of [4] Section 3 was formulated for \( \sigma(y) = y \). In order to discuss the precise relations to [4] the author has changed this into the slightly more general setting of \( \sigma(t, y) \).

The author would like to thank Walter Schachermayer, who gave the author the opportunity to stay in 1999 at the Technical University Vienna, where the present paper was written.

2. THE SETTING AND SOME PRELIMINARIES

The following notation and facts are used throughout this paper, also in Section 3, although the setting of this section is slightly more general. We fix a complete probability space \([\Omega, \mathcal{F}, \mathbb{P}]\), a time horizon \( T > 0 \), and a filtration \((\mathcal{F}_t)_{t \in [0, T]}\) satisfying the usual conditions (cf. [9] (p. 3)), which is generated by a Brownian motion \((W_t)_{t \in [0, T]}\) with \( W_0 \equiv 0 \). By \( S = (S_t)_{t \in [0, T]} \) we denote the geometric Brownian motion \( S_t \coloneqq \exp \left( W_t - \frac{1}{2} t \right) \) and let throughout this paper \( f : (0, \infty) \to [0, \infty) \) be a Borel measurable function such that

\[
  f(S_T) \in L_2(\Omega, \mathcal{F}, \mathbb{P}).
\]

To represent \( f(S_T) \) as a stochastic integral with respect to \((S_t)_{t \in [0, T]}\) we let

\[
  F(t, y) := \mathbb{E} f(yS_T) \quad \text{and} \quad \varphi(t, y) := \frac{\partial F}{\partial y}(t, y); \quad \text{note that} \quad F \in C^{\infty, \infty}([0, T] \times \mathbb{R}_+) \quad \text{according to Lemma A.2 from the appendix.}
\]

Then it is known that, on \((0, T) \times [0, \infty)\),

\[
  \frac{\partial F}{\partial t} + \frac{y^2}{2} \frac{\partial^2 F}{\partial y^2} = 0,
\]

see [8] (Corollary 5.1.3), and that

\[
  \mathbb{E} \left[ f(S_T) \mid \mathcal{F}_t \right] = F(t, S_t) = \mathbb{E} f(S_T) + \int_0^t \varphi(u, S_u) dS_u
\]

for \( t \in [0, T] \) holds a.s. (see [6] (Section 4.3)), where \( \mathbb{E} \left[ f(S_T) \mid \mathcal{F}_t \right] \) \( t \in [0, T] \) is always assumed to be càdlàg. The function \( H : [0, T) \to [0, \infty) \), defined by

\[
  H(u) := \left( \mathbb{E} \left[ S_u \frac{\partial \varphi}{\partial y}(u, S_u) \right] \right)^\frac{1}{2} \in C([0, T])
\]

will be the main tool in the consideration of our approximation problem (the continuity is proved in Lemma A.3 from the appendix). Finally, throughout this paper \( A \sim_e B \) (\( A, B \geq 0, c > 0 \)) is an abbreviation for \( \frac{A}{c} \leq B \leq cA \).

3. ONE STEP APPROXIMATION

Let us consider the stochastic differential equation

\[
  dX_t = \sigma(t, X_t) dW_t \quad \text{with} \quad X_0 = x_0 \in \mathbb{R}
\]

where the continuous function \( \sigma : [0, T] \times \mathbb{R} \to \mathbb{R} \) satisfies

\[
  |\sigma(t, y_1) - \sigma(t, y_2)| \leq K |y_1 - y_2| \quad \text{and} \quad |\sigma(t, y)| \leq K (1 + |y|)
\]
for $y_1, y_2, y \in \mathbb{R}$ and $t \in [0, T]$. Under these assumptions it is known that there is a unique solution $(X_t)_{t \in [0, T]}$ of (3), which can be assumed to be continuous for all $\omega \in \Omega$. Furthermore, let

$$A = \frac{\partial}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial y^2}.$$ 

The aim of this section is to prove

**Theorem 3.1.** Let $0 \leq a < b < T$ and $\Phi(t, y)(\omega) = [h(t, y) - v_u] \sigma(t, y)$ such that

(i) $h, \sigma \in C^{1,2}([0, T] \times \mathbb{R})$,

(ii) $\sup_{u \in [a, b]} |\Phi(u, X_u)| \in L_2(\Omega, \mathcal{F}, \mathbb{P})$,

(iii) $\left| \frac{\partial \Phi}{\partial y}(t, y) \right| \leq \kappa$ on $[0, T] \times \mathbb{R},$

(iv) $|A \Phi(t, y)| \leq \kappa |\Phi(t, y)|$ on $[0, T] \times \mathbb{R},$

(v) $v_u$ is an $\mathcal{F}_u$-measurable function taking a finite number of values, where $\kappa \geq 0$ is a fixed constant. Then one has

$$\int_a^b \mathbb{E} \Phi(t, X_t)^2 dt \sim C(\kappa) \int_a^b \mathbb{E} \left[ \Phi(a, X_a)^2 + [b - u] \left[ \frac{\sigma^2 \partial h}{\partial y} \right]^2 (u, X_u) \right] du$$

where $C : (0, T] \to [1, \infty]$ depends on $\kappa$ and $T$ only, is increasing, and satisfies $\lim_{u \to 0} C(\kappa) = 1$.

**Remark 3.2.** (i) Among other results (see for example [11] and [12]) the results of [4] suggest a close correspondence between the quantities $\int_a^b \mathbb{E} \Phi(t, X_t)^2 dt$ and $\int_a^b \mathbb{E} \left[ [b - u] \left[ \frac{\sigma^2 \partial h}{\partial y} \right]^2 (u, X_u) \right] du$. However, an equivalence like (4) is not proved there, although the authors use in their first step Itô’s formula in the same way as we do this here. The setting of [4] corresponds to:

(a) $v_u := h(a, X_a)$, so that $\Phi(a, X_a) = 0.$

(b) $g(X_t) = u + \int_0^t h(u, X_u) dX_u$ where $g$ is given by $g(y) = \chi_{[K, \infty)}(y)$ or $g(y) = (y - K)^+ \alpha$ with $0 < \alpha < 1/2$ and $K > 0$ (these $h$ satisfy the PDE in Example 3.6(ii) below).

(c) $\sigma(t, y) = \sigma(y)$ and $\sigma$ satisfies some technical assumptions.

Relation (4) simplifies and clarifies [4]. In particular it shows that it is not necessary to consider the error terms $A_2$ and $A_3$ from [4](Section 3.1).

(iii) The usage of the function $v_u$ instead of $\varphi(a, X_a)$ in (4) allows us to include the quantities $a^k_{\mathcal{F}}(\cdot, \tau)$ and $a^k_{\mathcal{F}}(\cdot)$ from Definition 4.1 below in our considerations.

Let us turn to the proof of Theorem 3.1.

**Lemma 3.3.** Let $0 \leq a \leq t < T$ and $\Phi(u, y)(\omega) = \sum_{i=1}^N \Phi(u, y) \chi_{A_i}(\omega)$, where

(i) $\Phi \in C^{1,2}([0, T] \times \mathbb{R})$,

(ii) $\sup_{u \in [a, t]} |\Phi(u, X_u)| \in L_2(\Omega, \mathcal{F}, \mathbb{P})$,

(iii) $\sup_{u \in [a, t]} (|\Phi|)(u, X_u) \in L_1(\Omega, \mathcal{F}, \mathbb{P})$,

(iv) $(A_1, \ldots, A_N)$ is an $\mathcal{F}_u$-measurable partition of $\Omega$.

Then one has

$$\mathbb{E} \Phi(t, X_t)^2 = \mathbb{E} \Phi(a, X_a)^2 + 2 \mathbb{E} \int_a^t (\Phi A \Phi)(u, X_u) du + \mathbb{E} \int_a^t \left[ \frac{\partial \Phi}{\partial y} \right]^2 (u, X_u) du.$$
Letting $P \mathcal{R}^o$. Applying Itô’s formula yields
\[
\Phi(t, X_t)^2 = \Phi(a, X_a)^2 + \int_a^t \left[ \Phi_y \left( \frac{\partial \Phi}{\partial y} \right) (u, X_u) du + \left( \sigma \frac{\partial \Phi}{\partial y} \right)^2 (u, X_u) \right] du \text{ a.s.}
\]

Letting
\[
\beta_n := \inf \left\{ \beta \in [a, t] \mid |(\Phi A \Phi)(\beta, X_\beta)| \vee \left| \Phi \left[ \frac{\partial \Phi}{\partial y} \right] (\beta, X_\beta) \right| \vee \int_a^\beta \left[ \Phi_y \left( \frac{\partial \Phi}{\partial y} \right) (u, X_u) du \right] \geq n \right\} \wedge t
\]
on one gets
\[
\mathbb{E} \Phi(\beta_n, X_{\beta_n})^2 = \mathbb{E} \Phi(a, X_a)^2 + \mathbb{E} \int_a^{\beta_n} \left[ 2(\Phi A \Phi)(u, X_u) + \left( \sigma \frac{\partial \Phi}{\partial y} \right)^2 (u, X_u) \right] du
\]
\[
= \mathbb{E} \Phi(a, X_a)^2 + \mathbb{E} \int_a^{\beta_n} \left[ 2(\Phi A \Phi)(u, X_u) \right] du + \mathbb{E} \int_a^{\beta_n} \left( \frac{\partial \Phi}{\partial y} \right)^2 (u, X_u) du.
\]

By $n \to \infty$ we can conclude the proof. \hfill \Box

**Lemma 3.4** (Gronwall, cf. [10]). Let $0 \leq a \leq t < \infty$ and let $f : [a, t] \to [0, \infty]$ be a measurable function such that $\int_a^t f(s) ds < \infty$ and
\[
f(u) \leq A + B \int_a^u f(s) ds \quad \text{for} \quad u \in [a, t],
\]
where $A, B \geq 0$ are fixed constants. Then one has $f(t) \leq Ae^{Bu}$.

**Lemma 3.5.** Let $0 \leq a \leq t \leq b < T$ and $\Phi(t, y)(\omega) = [h(t, y) - v_n] \sigma(t, y)$ such that
\begin{enumerate}[(i)]
\item $h, \sigma \in C^{1,2}([0, T] \times \mathbb{R})$,
\item $\sup_{u \in [a, b]} |\Phi(u, X_u)| \in L_2(\Omega, \mathcal{F}, \mathbb{P})$,
\item $\frac{\partial \Phi}{\partial y}(t, y) \leq \kappa_\sigma$ on $[0, T] \times \mathbb{R}$,
\item $|A \Phi(t, y)| \leq \kappa_{A, \Phi}(t, y)$ on $[0, T] \times \mathbb{R}$,
\item $v_n$ is an $\mathcal{F}_a$-measurable function taking a finite number of values,
\end{enumerate}
where $\kappa_\sigma, \kappa_{A, \Phi} \geq 0$ are fixed constants. Then
\[
\frac{1}{c} \mathbb{E} \Phi(t, X_t)^2 \leq \mathbb{E} \Phi(a, X_a)^2 + \int_a^t \mathbb{E} \left[ \sigma^2 \frac{\partial h}{\partial y} \right]^2 (u, X_u) du \leq \]
\[
d \left[ \mathbb{E} \Phi(t, X_t)^2 + \int_a^t \mathbb{E} \Phi(u, X_u)^2 du \right]
\]
with $c := 2e^{2T(\kappa_\sigma^2 + \kappa_{A, \Phi})}$ and $d := 4\kappa_{A, \Phi} + 2\kappa_\sigma^2$. Consequently,
\[
\int_a^b \mathbb{E} \Phi(t, X_t)^2 dt \prec C \left[ b - a \right] \mathbb{E} \Phi(a, X_a)^2 + \int_a^b \left[ b - u \right] \mathbb{E} \left[ \sigma^2 \frac{\partial h}{\partial y} \right]^2 (u, X_u) du
\]
with $C := c \vee (d[1 + b - a])$. 

Proof. First we remark that \[ \left( \frac{\partial \Phi}{\partial y} \right)^2 = \left( \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right)^2 \] so that
\[
\frac{1}{2} \left[ \sigma^2 \frac{\partial h}{\partial y} \right]^2 - \left( \frac{\partial \Phi}{\partial y} \right)^2 \leq \left( \sigma \frac{\partial^2 \Phi}{\partial y^2} \right)^2 \leq 2 \left[ \sigma^2 \frac{\partial h}{\partial y} \right]^2 + 2 \left( \frac{\partial \Phi}{\partial y} \right)^2.
\]

Now the right-hand side inequality of our assertion follows because of
\[
E \Phi(t, X_t)^2 \geq E \Phi(a, X_a)^2 - 2 \kappa A \int_a^t \Phi(u, X_u)^2 du + \frac{1}{2} \int_a^t E \left[ \sigma^2 \frac{\partial h}{\partial y} \right]^2 (u, X_u) du - \kappa_0^2 \int_a^t \Phi(u, X_u)^2 du.
\]

For the remaining inequality we observe that
\[
E \Phi(t, X_t)^2 \leq 2 \left[ E \Phi(a, X_a)^2 + \int_a^t E \left[ \sigma^2 \frac{\partial h}{\partial y} \right]^2 (u, X_u) du \right] + 2 \left[ \kappa_A + \kappa_0^2 \right] \int_a^t E \Phi(u, X_u)^2 du
\]
as a consequence of Lemma 3.3. Applying Lemma 3.4 gives
\[
E \Phi(t, X_t)^2 \leq 2 \left[ E \Phi(a, X_a)^2 + \int_a^t E \left[ \sigma^2 \frac{\partial h}{\partial y} \right]^2 (u, X_u) du \right] e^{2(\kappa_A + \kappa_0^2)t}
\]
so that we are done. The consequently-part follows by integration over \([a, b]\). \(\square\)

Proof of Theorem 3.1. Let \(0 < \varepsilon < 1\) and \((\frac{1}{\varepsilon} + 1) \kappa^2 + 2\kappa) [b-a] < 1\). By a simple computation, in particular using
\[
(1-\varepsilon)a^2 - \left( \frac{1}{\varepsilon} - 1 \right) b^2 \leq (a+b)^2 \leq (1+\varepsilon)a^2 + \left( 1 + \frac{1}{\varepsilon} \right) b^2,
\]
we derive from Lemma 3.3 the equivalence (4) with the multiplicative constant
\[
C(\varepsilon, b-a) := \frac{1 + \varepsilon}{1 - ((\frac{1}{\varepsilon} + 1) \kappa^2 + 2\kappa) [b-a]} \sqrt{1 + ((\frac{1}{\varepsilon} - 1) \kappa^2 + 2\kappa) [b-a]}.
\]
Defining
\[
C_0(\varepsilon) := C(\varepsilon, \varepsilon^2) \quad \text{for} \quad \left( \frac{1}{\varepsilon} + 1 \right) \kappa^2 + 2\kappa \varepsilon^2 < 1
\]
we obtain a continuous function satisfying \(\lim_{\varepsilon \to 0} C_0(\varepsilon) = 0\). Taking into account Lemma 3.5 we easily find the desired function \(C(\varepsilon)\). \(\square\)

Now let us consider the basic situations in which Theorem 3.1 applies.

Example 3.6. Let
\begin{enumerate}
\item \(\sigma(t, y) = \sigma(y)\);
\item \(\frac{\partial}{\partial y} (t, y) + \frac{\sigma^2(y)}{2} \frac{\partial^2}{\partial y^2} (t, y) + \sigma(y) \sigma'(y) \frac{\partial}{\partial y} (t, y) = 0\);
\item \(\| \sigma(y) \sigma''(y) \| \leq 2\kappa\).
\end{enumerate}
Then one has \(A\Phi = \left| \frac{\sigma \Phi}{\Phi} \right| \leq \kappa |\Phi|\).
Proof. We have

\[ A\Phi = \frac{\partial \Phi}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 \Phi}{\partial y^2} = \frac{\partial \Phi}{\partial t} + \frac{\sigma^2}{2} \left[ \frac{\partial^2 \Phi}{\partial y^2} + \frac{\sigma}{\partial y} \frac{\partial \Phi}{\partial y} + \left( h - \nu_u \right) \frac{\partial^2 \Phi}{\partial y^2} \right] \]

\[ = \sigma \left[ \frac{\partial \Phi}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 \Phi}{\partial y^2} + \sigma \frac{\partial \Phi}{\partial y} \right] + \frac{\sigma^2}{2} \left( h - \nu_u \right) \frac{\partial^2 \Phi}{\partial y^2} = \frac{\sigma^2}{2} \frac{\partial^2 \Phi}{\partial y^2} \Phi. \]

\[ \square \]

The prototype for Example 3.6(ii) can be found in (5) below. The next example concerns Theorem 3.1(ii).

Example 3.7. Let \( 0 \leq a < b < T \) and assume that \( \varphi \in C^{\infty,\infty}([0,T] \times (0,\infty)) \) is defined as in (and under the assumptions of) Section 2. If \( \Phi(u, y) := [\varphi(u, y) - \nu_u] y \), where \( \nu_u \) is an \( \mathcal{F}_u \)-measurable function taking a finite number of values, then one has

\[ \sup_{u \in [a, b]} \left| \Phi(u, X_u) \right| \in L_2(\Omega, \mathcal{F}, \mathbb{P}). \]

Proof. Because of \((a + b)^2 \leq 2(a^2 + b^2)\) and \( \mathbb{E} \sup_{u \in [0, T]} S_u^2 < \infty \) it is sufficient to show that \( \mathbb{E} \sup_{u \in [a, b]} [\varphi(u, S_u) S_u]^2 < \infty \). Taking \( 1 < p, q < \infty \) with \( 1 = \frac{1}{p} + \frac{1}{q} \) and \( 1 < p < 2 \), and using Lemma A.2, we have

\[ \mathbb{E} \sup_{u \in [a, b]} [\varphi(u, S_u) S_u]^2 = \mathbb{E} \sup_{u \in [a, b]} \left[ \mathbb{E} \left[ f \left( S_u \mathcal{F}_{T-u} \right) \frac{W_{T-u}}{T-u} \right] \right]^2 \]

\[ \leq \mathbb{E} \sup_{u \in [a, b]} \left[ \mathbb{E} f \left( S_u \mathcal{F}_{T-u} \right) \right]^\frac{p}{2} \left[ \mathbb{E} \left[ \frac{W_{T-u}}{T-u} \right]^q \right]^\frac{2}{p} \]

\[ \leq \left[ \mathbb{E} \left[ \frac{W_T}{T-b} \right]^\frac{p}{2} \right]^\frac{1}{2} \mathbb{E} \sup_{u \in [a, b]} \left[ \mathbb{E} f \left( S_u \mathcal{F}_{T-u} \right) \right]^\frac{p}{2} \]

\[ = \left[ \mathbb{E} \left[ \frac{W_T}{T-b} \right]^\frac{p}{2} \right]^\frac{1}{2} \mathbb{E} \sup_{u \in [a, b]} \left[ \mathbb{E} \left( f(S) \right)^p \left| \mathcal{F}_u \right) \right]^\frac{p}{2} \]

\[ \leq \left[ \mathbb{E} \left[ \frac{W_T}{T-b} \right]^\frac{p}{2} \right]^\frac{1}{2} \left[ \frac{2/p}{2/p - 1} \right]^\frac{1}{2} \mathbb{E} f(S)^2. \]

\[ \square \]

Combining Theorem 3.1, Example 3.6, Example 3.7, and

\[ \frac{\partial^2 \Phi}{\partial t^2} (t, y) + \frac{y^2}{2} \frac{\partial^2 \Phi}{\partial y^2} (t, y) + \frac{\partial \Phi}{\partial y} (t, y) = 0, \]

which follows from (1), yields to

(5)
Corollary 3.8. Let \(0 \leq a < b < T\) and let \(\varphi \in C^{\infty, \infty}[0, T) \times (0, \infty)\) and \(H : [0, T) \to [0, \infty)\) be given as in (and under the assumptions of) Section 2. Then
\[
\int_a^b \mathbb{E}[[\varphi(t, S_t) - \varphi(a, S_a)]^2] S_a^2 dt \sim C(T-a) \int_a^b [b-u]H(u)^2 du
\]
where \(C : (0, \infty) \to [1, \infty)\) is the function from Theorem 3.1 and the infimum is taken over all \(\mathcal{F}_a\)-measurable random variables \(v_a\).

Remark 3.9. Note that in Corollary 3.8 the range of definition of \(\varphi\) is \([0, T) \times (0, \infty)\) and not \([0, T) \times \mathbb{R}\). However, this does not make any difference in the proof, since Itô’s formula can be applied in Lemma 3.3 in this situation as well.

Exactly the same proof yields to the following refined version of Corollary 3.8, which is used in Section 6.

Corollary 3.10. Let \(0 \leq a < b < T\) and let \(\varphi \in C^{\infty, \infty}[0, T) \times (0, \infty)\) be given as in (and under the assumptions of) Section 2. Then a.s.
\[
\int_a^b \mathbb{E}_{\mathcal{F}_a}[\varphi(t, S_t) - \varphi(a, S_a)]^2 S_a^2 dt \sim C(T-a) \int_a^b [b-u] \mathbb{E}_{\mathcal{F}_a} \left[ \frac{S_a}{u} \frac{\partial}{\partial y} \right] (u, S_u) du.
\]
where \(C : (0, \infty) \to [1, \infty)\) is the function from Theorem 3.1.

4. Multi-step approximation

4.1. Basic definitions. From the one-step approximation considered in Corollary 3.8 we can immediately derive estimates for the multi-step approximation. To this end we use

Definition 4.1. For \(n \in \{1, 2, \ldots\}\) we let \(T_n\) be the set of all partitions \(\tau_n = (t_i)_{i=1}^n\) such that \(0 = t_0 \leq t_1 \leq \cdots \leq t_{n-1} < t_n = T\), \(\mathcal{T} := \bigcup_{n=1}^\infty T_n\), and \(\|t_i\|_{\infty} := \sup_{i=1, \ldots, n} |t_i - t_{i-1}|\). Moreover, for \(X = f(S_T) \in L_2(\Omega, \mathcal{F}, \mathbb{P})\), \(\tau = (t_i)_{i=0}^n \in \mathcal{T}\), and \(n \in \{1, 2, \ldots\}\) we let
\[
a(X; \tau) := \left( \sum_{i=1}^n \int_{t_{i-1}}^{t_i} [t_i - u]H(u)^2 du \right)^{1/2},
\]
\[
a^{\text{sim}}(X; \tau) := \left( \|X - \mathbb{E}X - \sum_{i=1}^n \varphi(t_{i-1}, S_{i-1}) [S_t - S_{t_{i-1}}] \|_{L_2} \right),
\]
\[
a^{KW}(X; \tau) := \inf \left\{ \left\| X - \mathbb{E}X - \sum_{i=1}^n v_{i-1} [S_t - S_{t_{i-1}}] \right\|_{L_2} \right\}.
\]
where the infimum is taken over all \( F_{n} \)-measurable \( v_{i} \) taking a finite number of values. Finally,

\[
\begin{align*}
\alpha_{n}(X) & := \inf \{ \alpha(X; \tau_{n}) \mid \tau_{n} \in \mathcal{T}_{n} \}, \\
\alpha_{n}^{\text{sim}}(X) & := \inf \{ \alpha^{\text{sim}}(X; \tau_{n}) \mid \tau_{n} \in \mathcal{T}_{n} \}, \\
\alpha_{n}^{\text{KW}}(X) & := \inf \{ \alpha^{\text{KW}}(X; \tau_{n}) \mid \tau_{n} \in \mathcal{T}_{n} \}.
\end{align*}
\]

**Remark 4.2.**  
(i) From Corollary 3.8 it follows that

\[
\int_{0}^{T} [T - u] H(u)^{2} du = \lim_{\delta \to 0} \int_{0}^{\delta} [b - u] H(u)^{2} du \leq C(T) \mathbb{E} \int_{0}^{T} \varphi(t, S_{t})^{2} dt < \infty.
\]

(ii) \( \alpha(X; \tau) \) is finite because of item (i).

(iii) \( \alpha^{\text{sim}}(X; \tau) \) is finite because, for \( f(S_{T}) \in L_{2}(\Omega, \mathcal{F}, \mathbb{P}) \) and \( a \in [0, T] \), one has

\[
\varphi(a, S_{a}) S_{a} \in L_{2}(\Omega, \mathcal{F}, \mathbb{P}),
\]

which implies \( \varphi(a, S_{a}) S_{a} \in L_{2}(\Omega, \mathcal{F}, \mathbb{P}) \) for \( 0 \leq a < b \leq T \). In fact, by Lemma A.2,

\[
\mathbb{E} \left[ \frac{\partial F}{\partial x}(a, S_{a}) S_{a} \right]^{2} = \mathbb{E} \left[ \frac{1}{T - a} \mathbb{E} [f(S_{a} S_{T-a}) W_{T-a}]^{2} \right] \\
\leq \mathbb{E} \left[ \frac{1}{T - a^{2}} \mathbb{E} [f(S_{a} S_{T-a})]^{2} \mathbb{E} [W_{T-a}]^{2} \right] \\
= \frac{1}{T - a} \mathbb{E} [f(S_{T})]^{2} < \infty.
\]

The rest follows by the independence of \( \varphi(a, S_{a}) S_{a} \) and \( \frac{S_{T}}{a} \).

(iv) \( \alpha^{\text{KW}}(X; \tau) \) is finite, since one can choose \( \nu_{T} = 0 \).

(v) The super-script "KW" should indicate Kunita–Watanabe, the super-script "sim" stands for simple.

### 4.2. Comparison of \( \alpha(X; \tau) \), \( \alpha^{\text{sim}}(X; \tau) \), and \( \alpha^{\text{KW}}(X; \tau) \).

From Corollary 3.8 we directly obtain

**Theorem 4.3.** For \( c = \sqrt{C(\|\tau\|_{\infty})} \) one has

\[
\frac{1}{c^{2}} \alpha^{\text{sim}}(X; \tau) \leq \frac{1}{c} \alpha(X; \tau) \leq \alpha^{\text{KW}}(X; \tau) \leq \alpha^{\text{sim}}(X; \tau).
\]

### 4.3. Comparison of \( \alpha_{n}(X) \), \( \alpha_{n}^{\text{sim}}(X) \), and \( \alpha_{n}^{\text{KW}}(X) \).

As main result we show

**Theorem 4.4.** For \( X = f(S_{T}) \in L_{2}(\Omega, \mathcal{F}, \mathbb{P}) \) the following holds.

(i) \( \lim_{n \to \infty} s_{n}(X) = 0 \) whenever \( s_{n} \in \{ \alpha_{n}, \alpha_{n}^{\text{sim}}, \alpha_{n}^{\text{KW}} \} \).

(ii) Fix \( n \in \{ 1, 2, \ldots \} \) and \( s_{n} \in \{ \alpha_{n}, \alpha_{n}^{\text{sim}}, \alpha_{n}^{\text{KW}} \} \). Then \( s_{n}(X) = 0 \) if and only if there exist constants \( c_{0}, c_{1} \geq 0 \) such that \( f(S_{T}) = c_{0} + c_{1} S_{T} \) a.s.

(iii) Assume that there do not exist constants \( c_{0}, c_{1} \geq 0 \) such that \( f(S_{T}) = c_{0} + c_{1} S_{T} \) a.s. Then

\[
\lim_{n \to \infty} \frac{\alpha_{n}^{\text{sim}}(X)}{\alpha_{n}(X)} = \lim_{n \to \infty} \frac{\alpha_{n}^{\text{KW}}(X)}{\alpha_{n}(X)} = 1.
\]
Remark 4.5. Theorem 4.3 and Theorem 4.4(ii) are linked to [12] (Théorème 2.1.2) and [4] (Theorem 2.1, Theorem 2.2) in the following way: Instead of considering the concrete orders of approximation we compare the approximation quantities \( a^{\text{sim}}(X; \tau_n) \) and \( a^{KW}(X; \tau_n) \) or \( a^{\text{sim}}(X) \) and \( a^{KW}(X) \) directly with the ‘geometric’ quantities \( a(X; \tau_n) \) and \( a_n(X) \), respectively. Here we are not restricted to equidistant nets of knots \( \{ t_1^p \}_{p=0}^\infty \) and, moreover, at this stage we obtain a statement which simultaneously holds true for all pay-off functions \( f \), as long as \( f(S_T) \in L_2(\Omega, \mathcal{F}, \mathbb{P}) \).

For the proof of Theorem 4.4 some preliminaries are needed.

Lemma 4.6. Fix \( \theta > 0 \) and assume that \( 0 \leq a_n < b_n \leq T \) with \( b_n - a_n = \theta \) and

\[
\lim_{n \to \infty} \int_{a_n}^{b_n} [b_n - u]H(u)^2 \, du = 0.
\]

Then there are \( T_0 \in (0, T) \) and \( c_0, c_1 \geq 0 \) such that

\[
\mathbb{E}(f(S_T) \mid \mathcal{F}_{T_0}) = c_0 + c_1 S_{T_0} \text{ a.s.}
\]

Proof. We can assume that \( \lim_n a_n = a_0 \) and \( \lim_n b_n = b_0 \) with \( 0 \leq a_0 < b_0 \leq T \) and \( b_0 - a_0 = \theta \). Then, by Fatou’s lemma,

\[
\int_{a_0}^{b_0} [b_0 - u]H(u)^2 \, du = \int_0^T [\chi_{[a_0, b_0]}(u) [b_0 - u]H(u)^2] \, du
\]

\[
\leq \int_0^T \liminf_{n \to \infty} [\chi_{[a_n, b_n]}(u) [b_n - u]H(u)^2] \, du
\]

\[
\leq \liminf_{n \to \infty} \int_0^T [\chi_{[a_n, b_n]}(u) [b_n - u]H(u)^2] \, du = 0
\]

so that, for \( u \in [a_0, b_0] \),

\[
\mathbb{E}\left[ \frac{\partial^2 F}{\partial y^2}(u, S_u) \right]^2 = H(u)^2 \geq 0 \quad \text{and} \quad \frac{\partial^2 F}{\partial y^2}(u, y) = 0
\]

for \( \lambda \)-almost \( y > 0 \). Hence, by continuity of \( F \),

(6) \[ \frac{\partial^2 F}{\partial y^2} = 0 \quad \text{on} \quad [a_0, b_0] \times (0, \infty). \]

Now fix an arbitrary \( T_0 \in (a_0, b_0) \). Then

\[
F(T_0, y) = c_0 + c_1 y
\]

for some \( c_0, c_1 \in \mathbb{R} \) as a consequence of (6). Because of

\[
c_0 + c_1 S_{T_0} = F(T_0, S_{T_0}) = \mathbb{E}(f(S_T) \mid \mathcal{F}_{T_0}) \geq 0 \text{ a.s.}
\]

we finally get \( c_0, c_1 \geq 0 \). □
Lemma 4.7. Let $T_0 \in (0, T)$ and $c_0, c_1 \geq 0$. If
\[
\mathbb{E} \left( f(S_T) \mid \mathcal{F}_{T_0} \right) = c_0 + c_1 S_{T_0} \text{ a.s.,}
\]
then $f(S_T) = c_0 + c_1 S_T$ a.s.

Proof. Let $f_0(y) := f(y) - c_0 - c_1 y$ for $y > 0$ and
\[
h(x) := f_0 \left( e^{x - \frac{x^2}{2}} \right) \quad \text{for} \quad x \in \mathbb{R}.
\]
Our assumption implies that \(\mathbb{E} \left( f \left( S_{T-T_0} S_{T_0} \right) \right) = c_0 + c_1 S_{T_0} \) a.s., so that
\[
\mathbb{E} \left( f \left( S_{T-T_0} y \right) \right) = c_0 + c_1 y \quad \text{and} \quad \mathbb{E} \left( f_0 \left( S_{T-T_0} y \right) \right) = 0 \quad \text{for all} \quad y > 0.
\]
Because of $h \left( (\log y + \frac{y^2}{2} + \frac{T}{T_0}) \right) = f_0 \left( S_{T-T_0} y \right)$ we obtain
\[
\mathbb{E} h \left( x + \frac{y^2}{2} \right) = 0 \quad \text{for all} \quad x \in \mathbb{R}.
\]
Differentiating with respect to $x$ yields, for $n = 0, 1, 2, \ldots$,
\[
0 = \mathbb{E} h \left( x + \sqrt{T - T_0} W_1 \right) H_n(W_1) = \int_{\mathbb{R}} h \left( x + \sqrt{T - T_0} \xi \right) \frac{dH_n(\xi)}{\sqrt{2\pi}}.
\]
Since
\[
\xi \to h \left( x + \sqrt{T - T_0} \xi \right) \in L_2 \left( \mathbb{R}, e^{-\frac{\xi^2}{2}} d\xi \right),
\]
which follows from $\int_{\mathbb{R}} h(\sqrt{T} \xi) e^{-\frac{\xi^2}{2}} \frac{d\xi}{\sqrt{2\pi}} = \mathbb{E} f(S_T)^2 < \infty$, and since the normalized Hermite polynomials form a complete orthonormal system we deduce that $h(\xi) = 0$ for $\lambda$-almost all $\xi \in \mathbb{R}$ which implies $f_0(y) = 0$ for $\lambda$-almost all $y > 0$.

\[\square\]

Proposition 4.8. For $X = f(S_T) \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ the following holds.
(i) One has $\lim_{n \to \infty} a_n(X) = 0$.
(ii) If there are not constants $c_0, c_1 \geq 0$ such that $f(S_T) = c_0 + c_1 S_T$ a.s., then
(a) $a_n(X) > 0$ for $n = 1, 2, \ldots$, and
(b) $\lim_{n \to \infty} \|a_n^\prime \|_\infty = 0$ whenever $\lim_{n \to \infty} a(X; \tau_n) = 0$ and $\tau_n \in \mathcal{T}_n$.

Proof. Assertion (i) follows from Lemma A.3 and Remark 4.2(i). Let us turn to assertion (ii). To verify part (a) fix $n \geq 1$ and assume that $a_n(X) = 0$. Then there is a sequence $\tau_n^{(i)} \in \mathcal{T}_n$ with $\lim_{n \to \infty} a(X; \tau_n^{(i)}) = 0$. This implies that there are $a_i, b_i \in [0, T]$ with $b_i - a_i = \frac{\theta}{n}$ and
\[
\lim_{n \to \infty} \int_{a_i}^{b_i} [b - u] H(u)^2 du = 0.
\]
Combining Lemma 4.6 and Lemma 4.7 leads to a contradiction to our assumptions on $f$ which proves part (a). To verify the remaining part (b) assume that $a(X; \tau_n) \to_n 0$ but $\limsup_{n \to \infty} \|a_n^\prime\|_\infty > 0$. Similarly we find $a_n, b_n \in [0, T]$ with $b_n - a_n = \theta$ and
\[
\lim_{n \to \infty} \int_{a_n}^{b_n} [b_n - u] H(u)^2 du = 0,
\]
and can conclude as in the proof of part (a). \[\square\]
Proof of Theorem 4.4. (a) Assertion (i) follows from Proposition 4.8(1) and Theorem 4.3.

(b) Assume that \( s_n(X) = 0 \). From Theorem 4.3 we get that \( a_n(X) = 0 \). Applying Proposition 4.8(ii)(a)] yields the existence of \( a_0, c_1 \geq 0 \) such that \( f(S_T) = c_0 + c_1 S_T \) a.s. Conversely, if \( f(S_T) = c_0 + c_1 S_T \) a.s then trivially \( s_n(X) = 0 \). Hence we have shown assertion (ii).

(c) In order to prove assertion (iii) we need to show

\[
\limsup_{n \to \infty} \frac{a_n(X)}{a_n^{KW}(X)} \leq 1 \quad \text{and} \quad \limsup_{n \to \infty} \frac{a_n^{\sim}(X)}{a_n(X)} \leq 1.
\]

Assume that we have shown inequalities (7). Then

\[
\limsup_{n \to \infty} \frac{a_n(X)}{a_n^{\sim}(X)} \leq \limsup_{n \to \infty} \frac{a_n(X)}{a_n^{KW}(X)} \leq 1
\]

and

\[
\limsup_{n \to \infty} \frac{a_n^{KW}(X)}{a_n(X)} \leq \limsup_{n \to \infty} \frac{a_n^{\sim}(X)}{a_n(X)} \leq 1
\]

as a consequence of \( a_n^{KW}(X) \leq a_n^{\sim}(X) \) which proves our third assertion. Let us turn to the inequalities (7). Because of \( a_n^{KW}(X) > 0 \) we find a sequence \( \tau_n \in \mathcal{T}_n \) such that \( (1 - \frac{1}{2n}) a_n^{KW}(X; \tau_n) \leq a_n^{KW}(X) \). Because of \( a_n^{KW}(X) \to_n 0 \) we necessarily have that \( \lim_{n \to \infty} \|\tau_n\|_{\infty} = 0 \) (see Proposition 4.8 and Theorem 4.3). Hence, by Theorem 4.3,

\[
\frac{a_n(X)}{a_n^{KW}(X)} \leq \frac{a(X; \tau_n)}{(1 - \frac{1}{2n}) a_n^{KW}(X; \tau_n)} \leq \frac{\sqrt{C(\|\tau_n\|_{\infty})}}{(1 - \frac{1}{2n})}
\]

which implies the first inequality of (7). To see the second one we find a sequence \( \sigma_n \in \mathcal{T}_n \) such that \( (1 - \frac{1}{2n}) a_n(X; \sigma_n) \leq a_n(X) \). Because of \( a_n(X) \to_n 0 \) Proposition 4.8 implies that \( \lim_{n \to \infty} \|\sigma_n\|_{\infty} = 0 \). Hence, again by Theorem 4.3,

\[
\frac{a_n^{\sim}(X)}{a_n(X)} \leq \frac{a_n^{\sim}(X; \sigma_n)}{(1 - \frac{1}{2n}) a_n(X; \sigma_n)} \leq \frac{\sqrt{C(\|\tau_n\|_{\infty})}}{(1 - \frac{1}{2n})}
\]

which implies the remaining inequality of (7).

\[\square\]

4.4. Convergence rate for \( \lim_{n \to \infty} a_n(X) = 0 \) and information about optimal approximation knots.

Definition 4.9. We let \( \overline{\Delta}(f(S_T)) := \sup \Delta \), where the supremum is taken over all \( \Delta \geq 0 \) such that there are \( c > 0 \) and \( \tau_n = (t_i^{(n)})_{i=0}^n \in \mathcal{T}_n \), \( n = 1, 2, \ldots \), such that

\[
a_n^{\sim}(f(S_T); \tau_n) \leq c a_n^{\sim}(f(S_T))
\]

for \( n = 1, 2, \ldots \) and

\[
\text{card} \left\{ 1 \leq i \leq n \mid t_i^{(n)} > [T - t] \right\} \leq c \left[ \frac{t}{T} \right] ^{\Delta} \quad \text{for} \quad t \in [0, T].
\]

In this subsection the following two theorems will be proved.
Theorem 4.10. Assume \( X = f(S_T) \in L_2(\Omega, \mathcal{F}, \mathbb{P}), 0 \leq \theta < 1 \), and \( c > 0 \) such that
\[
\left\| S_T \frac{\partial^2 \varphi}{\partial y^2}(t, S_t) \right\|_{L_2} \leq \frac{c}{|T - t|^\theta} \quad \text{for} \quad t \in [0, T).
\]
Then one has
\[
\sup_{n=1,2,\ldots} \sqrt{n} a_n(X) < \infty \quad \text{and} \quad \mathcal{D}(X) \geq 2[1 - \theta].
\]

Theorem 4.11. Assume \( X = f(S_T) \in L_2(\Omega, \mathcal{F}, \mathbb{P}), 0 \leq \theta < 1 \) and \( c > 0 \) such that
\[
\begin{align*}
(1) \quad & \frac{1}{c(T - t)^\theta} \leq \left\| S_T^2 \frac{\partial^2 \varphi}{\partial y^2}(t, S_t) \right\|_{L_2} \quad \text{for} \quad t \in [0, T), \\
(2) \quad & a_n(X) \leq \frac{c}{\sqrt{n}} \quad \text{for} \quad n = 1, 2, \ldots.
\end{align*}
\]
Then one has
\[
\inf_{n=1,2,\ldots} \sqrt{n} a_n(X) > 0 \quad \text{and} \quad \mathcal{D}(X) \leq 2[1 - \theta].
\]

Motivated by Theorems 4.10 and 4.11 and the consideration of the European–Call and Binary Option below we use

Definition 4.12. For \( 0 \leq \theta < 1 \) a random variable \( f(S_T) \in L_2(\Omega, \mathcal{F}, \mathbb{P}) \) belongs to \( S(\theta) \) provided that there is a constant \( c > 0 \) such that
\[
\frac{1}{c(T - t)^\theta} \leq \left\| S_T^2 \frac{\partial^2 \varphi}{\partial y^2}(t, S_t) \right\|_{L_2} \leq \frac{c}{|T - t|^\theta} \quad \text{for} \quad t \in [0, T).
\]

Now from Theorems 4.10 and 4.11 we immediately obtain

Corollary 4.13. Assume \( 0 \leq \theta < 1 \) and \( X = f(S_T) \in S(\theta) \). Then
\[
\mathcal{D}(X) = 2[1 - \theta] \quad \text{and} \quad \frac{1}{c \sqrt{n}} \leq a_n(X) \leq \frac{c}{\sqrt{n}}
\]
for \( n = 1, 2, \ldots \), where \( c > 0 \) is a constant depending on \( X \) only.

Now let us turn to the proof of Theorems 4.10 and 4.11.

Lemma 4.14. Let \( \Phi : [0, T] \to [0, \infty) \) be continuous such that
\[
\begin{align*}
(1) \quad & \Phi(u) > 0 \text{ on } [0, T], \\
(2) \quad & \frac{\Phi(u)}{u} \text{ is decreasing on } [0, T], \\
(3) \quad & \Phi(u) \text{ is decreasing on } [0, T], \\
(4) \quad & A := \int_0^T \frac{du}{\Phi(u)} < \infty.
\end{align*}
\]
Then, for all Borel measurable \( F : [0, T] \to [0, \infty) \) and \( 0 = t_0 \leq t_1 \leq \cdots \leq t_n = T \) such that
\[
B := \int_0^T \Phi(u) F(u) du < \infty \quad \text{and} \quad \int_0^{t_k} \Phi(u) F(u) du = \frac{k}{n} B,
\]
one has the inequality
\[
\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left| t_i - u \right| F(u) du \leq \frac{AB}{n}.
\]
Remark 4.15. The basic difference to the corresponding [4] (Lemma 3.4) and [12] (Lemme 2.2.20) is that we are looking for optimal time knots \( t_0, \ldots, t_n \) and not for the rate of convergence of the equidistant nets.

Proof Lemma 4.14. First we observe that (2) implies that \( u \to \frac{t_i - u}{\Phi(u)} \) is decreasing on \([0, b]\) for all \( 0 \leq b < T \). Now we get

\[
\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left[ t_i - u \right] F(u) du = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left[ \frac{t_i - u}{\Phi(u)} \right] \Phi(u) F(u) du \\
\leq \frac{B}{n} \sum_{i=1}^{n} \sup_{u \in [t_{i-1}, t_i]} \left[ \frac{t_i - u}{\Phi(u)} \right] = \frac{B}{n} \sum_{i=1}^{n} \frac{t_i - t_{i-1}}{\Phi(t_{i-1})} \leq \frac{B}{n} \int_{0}^{T} \frac{du}{\Phi(u)} = \frac{AB}{n}
\]

and we are done. \( \square \)

Proposition 4.16. Let \( \alpha \in (1, \infty) \), \( \theta \in [0, 1) \),

\[
\Phi(u) := \left[ \alpha + \log \left( 1 + \frac{1}{T - u} \right) \right]^{\alpha} [T - u], \quad \text{and} \quad F(u) := \frac{1}{(T - u)^{2\theta}}.
\]

Then assertions (i)–(iv) of Lemma 4.14 are satisfied. Moreover, if \( t_0, \ldots, t_n \) are defined as in Lemma 4.14, then for all \( 0 \leq \Delta < 2(1 - \theta) \) there is a constant \( c > 0 \), depending at most on \((\alpha, T, \Delta, \theta)\), such that

\[
(8) \quad \frac{1}{n} \text{card} \left\{ 1 \leq i < n \mid t_i > [T - \Delta] \right\} \leq c \left[ \frac{t}{T} \right]^{\Delta} \quad \text{for} \quad t \in [0, T].
\]

Proof. Assertions (i)–(iv) of Lemma 4.14 are obvious. To show inequality (8) we assume, for convenience, that \( 2(1 - \theta) - \Delta < 1 \) and \( \Delta > 0 \). Let \( \varepsilon \in (0, 1) \) be given by \( \varepsilon := 2(1 - \theta) - \Delta \) there is a constant \( c = c(\alpha, T, \varepsilon) > 0 \) such that

\[
\Phi(u) \leq c [T - u]^{1-\varepsilon} \quad \text{for} \quad u \in [0, T].
\]

Hence

\[
[T - t_i]^{\Delta} = \Delta \int_{t_i}^{T} [T - u]^{\Delta - 1} du \geq \frac{\Delta}{c} \int_{t_i}^{T} \Phi(u) F(u) du = \int_{t_i}^{T} \frac{du}{\Phi(u)} = \frac{n - i}{dn},
\]

with \( d = d(\alpha, T, \Delta, \varepsilon) := \frac{e}{\Delta T} \), and

\[
\frac{1}{n} \text{card} \left\{ 1 \leq i < n \mid t_i > [T - t] \right\} = \frac{1}{n} \text{card} \left\{ 1 \leq i < n \mid t_i > [T - t_i]^{\Delta} \right\} \leq \frac{1}{n} \text{card} \left\{ 1 \leq i < n \mid t_i > \frac{n - i}{dn} \right\} \leq \frac{1}{n} \text{card} \left\{ 1 \leq i < n \mid n - i < nt_\Delta \right\} \leq d t_\Delta = d T^{\Delta} \left[ t_\Delta \right]^{\Delta} \quad \square
\]
Proposition 4.17. Let $0 \leq \theta < 1$ and let $0 = t^{(n)}_0 \leq \cdots \leq t^{(n)}_n = T$, $n = 1, 2, \ldots$, be given such that
\[
\sum_{j=1}^{n} \int_{t^{(n)}_{j-1}}^{t^{(n)}_j} \frac{t^{(n)}_j - u}{T - u} \, du \leq \frac{c}{n}
\]
for some $c > 0$. Then one has, for $t \in [0, T]$,
\[
\lim\inf_{n \to \infty} \frac{1}{n} \operatorname{card} \left\{ 1 \leq i < n \mid t^{(n)}_i > |T - t| \right\} \geq \min \left\{ 1, \frac{1}{2c} \left( 2^{(1-\theta)} \right) \right\}.
\]

Proof. First we observe that
\[
\frac{c}{n} \geq \sum_{j=1}^{n} \int_{t^{(n)}_{j-1}}^{t^{(n)}_j} \frac{t^{(n)}_j - u}{T - u} \, du \geq \sum_{j=1}^{n} \left[ T - t^{(n)}_{j-1} \right]^{-2\theta} \frac{\left[ t^{(n)}_j - t^{(n)}_{j-1} \right]^2}{2}.
\]
Hence, for $1 \leq i < n$,
\[
\frac{2c}{n} \geq \left[ T - t^{(n)}_i \right]^{-2\theta} \sum_{j=i+1}^{n} \left[ t^{(n)}_j - t^{(n)}_{j-1} \right]^2 \geq \frac{\left[ T - t^{(n)}_i \right]^{2(1-\theta)}}{n-i},
\]
where we used
\[
|T - t^{(n)}_i| = \sum_{j=i+1}^{n} |t^{(n)}_j - t^{(n)}_{j-1}| \leq \sqrt{n-i} \sqrt{\sum_{j=i+1}^{n} \left[ t^{(n)}_j - t^{(n)}_{j-1} \right]^2}.
\]
Consequently,
\[
\frac{1}{n} \operatorname{card} \left\{ 1 \leq i < n \mid t^{(n)}_i > |T - t| \right\} \geq \frac{1}{n} \operatorname{card} \left\{ 1 \leq i < n \mid \frac{2^{(1-\theta)}}{n} > \frac{2c(n-i)}{n} \right\} \geq \min \left\{ \frac{n-1}{n}, \frac{2^{(1-\theta)}}{2c} - \frac{1}{n} \right\}
\]
which implies our assertion. \hfill \Box

5. Examples

Here we treat the basic examples, the pay-off functions of the European-Call Option $(S_T - K)^+$ and the pay-off function of the Binary Option $\chi_{(K, \infty)}(S_T)$.

Theorem 5.1. Let $K > 0$. Then one has
\begin{enumerate}
\item $(S_T - K)^+ \in \mathcal{S} \left( \frac{1}{4} \right)$,
\item $\chi_{(K, \infty)}(S_T) \in \mathcal{S} \left( \frac{1}{2} \right)$.
\end{enumerate}
Remark 5.2.  
(i) Theorem 5.1 is known from [12] and [4], although the notation $S(\theta)$ is not introduced.  
(ii) In [4] it is also shown that $((S_T - K)^+)^\alpha \in \mathcal{S} \left(\frac{3}{4} - \frac{\alpha}{2}\right)$ for $\alpha \in (0, \frac{1}{2})$.

Lemma 5.4 and Lemma 5.6 below imply Theorem 5.1 immediately. We state these lemmas because they are needed in this form in Section 6 below.

5.1. European–Call Option. Let $f(y) := (y - K)^+$. A simple computation shows

Lemma 5.3. For $(t, y) \in (0, T) \times (0, \infty)$ and $B := \frac{T}{T} - \log K$ one has

\[
\begin{align*}
\frac{\partial F}{\partial y}(t, y) &= N \left( \frac{\log y + B - \frac{1}{2}}{\sqrt{T - t}} \right), \\
\frac{\partial^2 F}{\partial y^2}(t, y) &= \frac{1}{y \sqrt{T - t}} N'( \frac{\log y + B - \frac{1}{2}}{\sqrt{T - t}} ),
\end{align*}
\]

where $N'$ is the derivative of $N$.

Lemma 5.4. Let $X = (S_T - K)^+$ or $X = (K - S_T)^+$, where $K > 0$. Then for $(t_0, y_0) \in (0, T) \times (0, \infty)$ or $(t_0, y_0) = (0, 1)$, and $t \in [t_0, T)$ one has a.s.

\[
\mathbb{E}_{S_{t_0} = y_0} \left[ S_t^2 \frac{\partial^2 F}{\partial y^2}(t, S_t) \right]^2 = \frac{K^2}{2\pi \sqrt{T - t} \sqrt{T + t - 2t_0}} e^{-\frac{(\log y_0 + \log(K+y_0))^2}{2\pi^2}}.
\]

Consequently, for $t_0 \in (0, T)$ one has

\[
\sup_{y_0 > 0} \mathbb{E}_{S_{t_0} = y_0} \left[ S_t^2 \frac{\partial^2 F}{\partial y^2}(t, S_t) \right]^2 = \frac{K^2}{2\pi \sqrt{T - t} \sqrt{T + t - 2t_0}}.
\]

Proof. Since the expressions $\frac{\partial F}{\partial y}$ for the European–Call Option $(S_T - K)^+$ and for the European–Put Option $(K - S_T)^+$ coincide it is sufficient to treat the first case only. Here one has, with $B := \frac{T}{T} - \log K$,

\[
\mathbb{E}_{S_{t_0} = y_0} \left[ S_t^2 \frac{\partial^2}{\partial y^2}(t, S_t) \right]^2 = \mathbb{E} \left[ \frac{\partial}{\partial y} S_{t_0}^2 \frac{\partial^2}{\partial y^2} (t, y_0, S_{t_0} - T_0) \right]^2
\]

\[
= \mathbb{E} \left[ \frac{y_0 S_{t_0} - T_0}{\sqrt{T - t}} N' \left( \frac{\log(y_0 S_{t_0} - T_0) + B - \frac{1}{2}}{\sqrt{T - t}} \right) \right]^2
\]

\[
= \frac{y_0^2}{[T - t][2\pi]} \mathbb{E} \left[ S_{t_0} - t_0 e^{-\frac{1}{2} \left( \frac{\log(y_0 + \log(K+y_0))^2}{\sqrt{T - t}} \right)^2} \right]^2
\]

\[
= \frac{e^{T - 2B}}{2\pi \sqrt{T - t} \sqrt{T + t - 2t_0}} e^{-\frac{(\log y + \log(K+y))^2}{2\pi^2}}. \quad \Box
\]
5.2. Binary Option. We let \( f(y) := \chi_{[K, \infty)}(y) \). Simple computations lead to the following lemmata.

**Lemma 5.5.** For \((t, y) \in (0, T) \times (0, \infty)\) and \(B := \frac{T}{T-t} + \log K\) one has

\[
F(t, y) = N \left( \frac{\log y - B + \frac{1}{2}}{\sqrt{T-t}} \right),
\]

\[
y \frac{\partial F}{\partial y}(t, y) = \frac{1}{\sqrt{T-t}} N \left( \frac{\log y - B + \frac{1}{2}}{\sqrt{T-t}} \right),
\]

\[
y^2 \frac{\partial^2 F}{\partial y^2}(t, y) = -\frac{1}{(T-t)^2} N \left( \frac{\log y - B + \frac{1}{2}}{\sqrt{T-t}} \right) \left[ 1 + \frac{\log y - B + \frac{1}{2}}{T-t} \right].
\]

**Lemma 5.6.** Let \(t_0 \in (0, T)\) and \(y_0 > 0\) or \(t_0 = 0\) and \(y_0 = 1\). Then one has, for \(t_0 \leq t < T\),

\[
\mathbb{E}_{S_{t_0}=y_0} \left[ S_t^2 \frac{\partial^2}{\partial y^2}(t, S_t) \right]^2 = \frac{1}{2\pi(\sqrt{T-t} + \sqrt{T-t_0})} \times
\]

\[
\left( \frac{t - t_0}{t + T - 2t_0} \right) \left[ 1 - \frac{t(T-t_0) + \log K_{y_0}}{t + T - 2t_0} \right] \left( 1 - \frac{[t(T-t_0) + \log K_{y_0}]^2}{(t + T - 2t_0)^2} \right).
\]

6. The approximation error of the European--Call and Put Option measured in \(\text{HMO}_2\)

We start with the definition of the cost--process \(C(\tau) = (C(\tau))_{\tau \in [0, T]}\) by which one can measures the approximation error more precisely than \(\varepsilon_{\text{sim}}(\cdot; \tau)\) or \(a_{\text{sim}}(\cdot)\) do this.

**Definition 6.1.** For \(\tau = (t_i)_{i=0}^n \in \mathcal{T}\) and \(t \in [0, T]\) we let

\[
C_t(\tau) := \mathbb{E} (f(S_T) \mid \mathcal{F}_t) - \mathbb{E} f(S_T) - \int_0^t \left[ \sum_{i=1}^n \chi_{(t_{i-1}, t_i]}(u) \varphi(t_{i-1}, S_{t_{i-1}}) \right] dS_u.
\]

The approximation error will be measured by the following \(\text{HMO}_2\)--quantity.

**Definition 6.2.** Let \(Y = (Y_t)_{t \in [0, T]}\) be ciddig and adapted and let \(\Phi = (\Phi_t)_{t \in [0, T]}\) be ciddig and adapted with \(\Phi_t(\omega) > 0\) for all \(\omega \in \Omega\) and \(t \in [0, T]\). Then we let

\[
||Y||_{\text{HMO}_2 \gamma} := \inf c, \text{ where the infimum is taken over all } c > 0 \text{ such that}
\]

\[
\mathbb{E} \left( \sup_{\tau \in [0, T]} \left| Y_{\tau} - Y_{\tau^-} \right|^2 \right) \leq c^2 \Phi_{\tau}^2 \text{ a.s.}
\]

for all stopping times \(0 \leq \tau \leq T\).

The aim of this section is to prove

**Theorem 6.3.** Let \(X = (S_T - K)^+\) or \(X = (K - S_T)^+\), where \(K > 0\). Then there exists a constant \(c > 0\), depending at most on \(T\) and \(K\), such that

\[
\frac{1}{c \sqrt{n}} \geq ||C(t_n^0)||_{\text{HMO}_2 \gamma} \leq \frac{c}{\sqrt{n}} \text{ for } n = 1, 2,\ldots
\]

and \(t_n^0 := \{0, \frac{T}{n}, \frac{2T}{n}, \ldots, \frac{T}{n}, T\},\)
Remark 6.4. Because of
\[ a_n^{\text{sim}}(X) \leq \| C_0(\tau_n^0) \|_{L_2} \leq \| C(\tau_n^0) \|_{\text{BMO}^c}. \]

Theorem 6.3 is stronger than the assertion \( a_n^{\text{sim}}(X) \sim \frac{1}{\sqrt{n}} \) proved before.

Let us turn to the proof of Theorem 6.3.

Lemma 6.5. For \( X = f(S_T) \in L_2(\Omega, \mathcal{F}, \mathbb{P}) \) let \( \varphi_i^* := \sup_{t \in [0, \tau_i]} \varphi(u, S_u) \) \( (i \in [0, T]). \) Then, for \( \tau = (t_i)_{i=0}^n \in \mathcal{T} \) and \( t_0 \leq a < t_n, \) one has a.s.

\[
\left( \mathbb{E} \left( |C_T(\tau) - C_a(\tau)|^2 \bigg| \mathcal{F}_a \right) \right)^{\frac{1}{2}} \leq 2e^{T} \left( \sqrt{t_0 - a} \varphi_i^* S_u + \right. \\
+ \left. \left( \int_a^{t_0} \left[ t_0 - u \right] \mathbb{E}_{\mathcal{F}_a} \left[ S_u \frac{\partial \varphi}{\partial y}(u, S_u) \right]^2 du + \sum_{i=0}^n \int_{t_{i-1}}^{t_i} \left[ t_i - u \right] \mathbb{E}_{\mathcal{F}_a} \left[ S_u \frac{\partial \varphi}{\partial y}(u, S_u) \right]^2 du \right)^{\frac{1}{2}} \right),
\]

where \( \sum_{i=0}^n \) is treated as zero.

Proof. One has, a.s.,

\[
\mathbb{E}_{\mathcal{F}_a} \left[ |C_T(\tau) - C_a(\tau)|^2 \right]^{\frac{1}{2}} = \left[ \mathbb{E}_{\mathcal{F}_a} \left[ \int_{(a, T]} \left[ \varphi(u, S_u) - \sum_{i=0}^n \chi_{(t_{i-1}, t_i]}(u) \varphi \left( t_{i-1}, S_{t_{i-1}} \right) \right] dS_u \right]^2 \right]^{\frac{1}{2}} \\
\leq \left[ \mathbb{E}_{\mathcal{F}_a} \left[ \int_{(a, t_0]} \left[ \varphi(a, S_u) - \varphi \left( t_{i_0-1}, S_{t_{i_0-1}} \right) \right] dS_u \right]^2 \right]^{\frac{1}{2}} + \\
+ \left[ \mathbb{E}_{\mathcal{F}_a} \left[ \int_{(a, t_0]} \left[ \varphi(u, S_u) - \varphi(a, S_u) \right] dS_u \right] \right]^{\frac{1}{2}} + \\
+ \sum_{i=0}^n \int_{t_{i-1}}^{t_i} \left[ \varphi(u, S_u) - \varphi \left( t_{i-1}, S_{t_{i-1}} \right) \right] dS_u \right]^{\frac{1}{2}} \\
=: E_1 + E_2.
\]

For \( E_1 \) one gets

\[
E_1 = \left[ \mathbb{E}_{\mathcal{F}_a} \left[ \left( \varphi(a, S_a) - \varphi \left( t_{i_0-1}, S_{t_{i_0-1}} \right) \right]^2 \left[ S_{t_{i_0}} - S_{t_0} \right]^2 \right] \right]^{\frac{1}{2}} \leq 2\varphi_i^* S_{t_{i_0}} \sqrt{t_{i_0} - a}.
\]

Using Corollary 3.10 one can estimate \( E_2 \) as

\[
E_2 \leq \sqrt{2e^{2T}} \left( \int_a^{t_0} \left[ t_0 - u \right] \mathbb{E}_{\mathcal{F}_a} \left[ S_u \frac{\partial \varphi}{\partial y}(u, S_u) \right]^2 du \right) +
\]
so that we are done.

Proof of Theorem 6.3. In view of Lemma 6.5, the fact that \( \varphi^*_n \leq 1 \) under the assumptions of Theorem 6.3, and in view of Lemma 5.4 it is sufficient to show that

\[
\frac{1}{\sqrt{T-a}} \int_{t_0}^T \frac{t_0-u}{\sqrt{T-u}} du + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \frac{t_i-u}{\sqrt{T-u}} du \leq \frac{c}{n}
\]

for \( t_i = \frac{i}{n} T \) which follows by a simple computation.

Remark 6.6. (i) For the Binary Option \( X = \chi_{[K, \infty)}(S_T) \), where \( K > 0 \), an estimate like Theorem 6.3 cannot be true because of the following reason: From Lemma 5.6 it follows, for \( t_0 \in (0, T) \), that

\[
\sup_{y_0 > 0} \mathbb{E}_{S_{t_0}-y_0} \left[ \frac{\partial^2 \mathcal{L}}{\partial y^2}(t, S_t) \right]^2 \lesssim \frac{1}{\sqrt{T-t_0} |T-t_0|^2},
\]

where \( c > 0 \) depends on \( K \) and \( T \) only. Consequently,

\[
\sup_{y_0 > 0} \int_{t_0}^T \left[ T-u \mathbb{E}_{S_{t_0}-y_0} \left[ \frac{\partial^2 \mathcal{L}}{\partial y^2}(u, S_u) \right] \right]^2 du \lesssim \frac{d}{1}
\]

with \( d > 0 \) depending on \( K \) and \( T \) only (in particular, one gets \( \frac{1}{n} \) for the lower estimate).

(ii) The situation is much worse for \( X = S_T^p \) with \( 0 < p < \infty \) and \( p \neq 1 \). Here one gets

\[
\sup_{y_0 > 0} \int_{t_0}^T \left[ T-u \mathbb{E}_{S_{t_0}-y_0} \left[ \frac{\partial^2 \mathcal{L}}{\partial y^2}(u, S_u) \right] \right]^2 du = \infty.
\]

Assume now that \( \hat{\mathbb{P}} \) is a probability measure on \( (\Omega, \mathcal{F}, \mathbb{P}) \) equivalent to \( \mathbb{P} \).

Lemma 6.7. Assume \( d > 0 \), \( Y = (Y_t)_{t \in [0, T]} \) to be càdlàg and adapted, and \( \Phi = (\Phi_t)_{t \in [0, T]} \) to be càdlàg and adapted, such that \( \Phi_t(\omega) > 0 \) for all \( \omega \in \Omega \) and \( t \in [0, T] \).

Assume that \( 0 < \theta < 1 \), \( \Phi^*_r < \infty \) a.s., and that

\[
\mathbb{P}_{\hat{\mathbb{P}}}
\left( \sup_{u \in [\tau, T]} \left| Y_u - Y_{\tau^-} \right| > \nu \big| \mathcal{F}_\tau \right) \leq \theta + \mathbb{P}_{\hat{\mathbb{P}}}
\left( \sup_{u \in [\tau, T]} \Phi_u > \frac{\nu}{d} \big| \mathcal{F}_\tau \right) \quad \text{a.s.}
\]

for all stopping times \( \tau: \Omega \to [0, T] \). Then one has the following:

(i) For all \( 0 < r < \infty \) there is a \( c_r = c_r(\theta, d) > 0 \) such that \( \sup_{1 \leq r < \infty} c_r(\theta, d) < \infty \) and, for all \( t \in [0, T] \),

\[
\mathbb{E}_{\hat{\mathbb{P}}}
\left( \sup_{u \in [t, T]} \left| Y_u - Y_{t^-} \right| \big| \mathcal{F}_t \right) \leq c_r \mathbb{E}_{\hat{\mathbb{P}}}
\left( \sup_{u \in [t, T]} \Phi_u \big| \mathcal{F}_t \right) \quad \text{a.s.}
\]

(ii) There exists constants \( \beta, b > 0 \), depending on \( \theta \) and \( d \) only, such that, for all \( t \in [0, T] \) and all \( \mu, \nu > 0 \), one has

\[
\mathbb{P}_{\hat{\mathbb{P}}}
\left( \sup_{u \in [t, T]} \left| Y_u - Y_{t^-} \right| > \nu \big| \mathcal{F}_t \right) \leq e^{1-\mu} + \beta \mathbb{P}_{\hat{\mathbb{P}}}
\left( \sup_{u \in [t, T]} \Phi_u > \frac{\nu}{b} \big| \mathcal{F}_t \right) \quad \text{a.s.}
\]
Lemma 6.7 can be proved along the lines of [3], [1], and [2].

Remark 6.8. If the process \( \log(\Phi_t) \) is a Lévy process under \( \mathbb{P} \), then one has a.s.

\[
\mathbb{E}_\mathbb{P} \left( \sup_{u \in [t,T]} \Phi_u \mid \mathcal{F}_t \right) = \mathbb{P}(X_t \in \cdot \mid \mathcal{F}_t) \mathbb{P} \left( \sup_{u \in [t,T]} \Phi_u \right) \leq \mathbb{P}((\Phi_T)^+) \mathbb{P} \left( \sup_{u \in [t,T]} \Phi_u \right),
\]

so that Lemma 6.7(i) turns into

\[
\mathbb{E}_\mathbb{P} \left( \sup_{u \in [t,T]} \Phi_u \mid \mathcal{F}_t \right) \leq \mathbb{P}((\Phi_T)^+) \mathbb{P} \left( \sup_{u \in [t,T]} \Phi_u \right),
\]

Lemma 6.9. Let \( d\mathbb{P} = Ld\mathbb{P} \) with

\[
(10) \quad (\mathbb{E} (L^r \mid \mathcal{F}_t))^\frac{1}{r} \leq c \mathbb{E} (L \mid \mathcal{F}_t) \text{ a.s. and } L(\omega) > 0
\]

for all \( \omega \in \Omega \) and all stopping times \( \tau : \Omega \to [0,T] \), where \( 1 < r < \infty \) and \( c > 0 \) are fixed. Then \( ||Y||_{\text{BM}^r} = 1 \) implies inequality (9), where, given \( \theta \), the constant \( d \) depends at most on \( r, \theta, \) and \( c \).

Summarizing the considerations above one gets

Corollary 6.10. Let \( d\mathbb{P} = Ld\mathbb{P} \), where \( L \) satisfies the reverse Hölder inequality (10), and let \( (C_t(\tau_n^0))_{t \in [0,T]} \) be as in Theorem 6.3. Then the following holds true.

(i) For all \( 0 < r < \infty \) and \( t \in [0,T] \) one has

\[
\mathbb{E}_\mathbb{P} \left( \sup_{u \in [t,T]} \left| C_u(\tau_n^0) - C_t(\tau_n^0) \right|^r \mid \mathcal{F}_t \right) \leq \left[ \frac{c_r}{\sqrt{n}} \right]^r \mathbb{E}_\mathbb{P} \left( \sup_{u \in [t,T]} S_u^r \mid \mathcal{F}_t \right) \text{ a.s.,}
\]

where \( c_r > 0 \) depends at most on \( T, K, v, \) and \( r \).

(ii) There exists constants \( \beta, b > 0, \) depending at most on \( T, K, v, \) and \( r, \) such that, for all \( t \in [0,T] \) and all \( \mu, v > 0 \), one has a.s.

\[
\mathbb{P} \left( \sup_{u \in [t,T]} \left| C_u(\tau_n^0) - C_t(\tau_n^0) \right| > \mu v \mid \mathcal{F}_t \right) \leq e^{1-\mu} + \beta \mathbb{P} \left( \sup_{u \in [t,T]} \Phi_u > \frac{\mu v}{\sqrt{n}} \mid \mathcal{F}_t \right).
\]

Appendix A. Some known facts

Here we summarize statements, which are either known or are of technical nature.

Lemma A.1 ([5](Section 4.3)). Let \( h : \mathbb{R} \to \mathbb{R} \) be Borel-measurable and \( \theta > 0 \). Assume that

\[
\int_{\mathbb{R}} e^{-|x|^\theta} |h(x)| dx < \infty.
\]

Then \( \psi(s,x) := \mathbb{E} [h(x + W_s)] \) exists (and is finite) for \( (s,x) \in (0, \frac{1}{\sqrt{2}}) \times \mathbb{R} \) and has partial derivatives of all orders. In particular, for \( n = 1, 2, ..., \)

\[
\frac{\partial^n \psi}{\partial x^n}(s,x) = \mathbb{E} \left[ h(x + \sqrt{s}W_1) H_n(W_1) \right],
\]

where \( (H_n)_{n=1}^{\infty} \) is the sequence of Hermite polynomials \( H_1(x) = x, H_2(x) = x^2 - 1, ... \).
Given a Borel measurable function $f : (0, \infty) \to [0, \infty)$ and $(t, y) \in [0, T] \times (0, \infty)$ we let throughout this appendix
\[ F(t, y) := \mathbb{E} f(y S_t) \in [0, \infty] \quad \text{where} \quad t^* := T - t. \]

**Lemma A.2** (cf. [8](Section 5.1.5)). Let $f(S_T) \in L_p(\Omega, \mathcal{F}, \mathbb{P})$ for some $1 < p < \infty$. Then one has the following.

(i) $F \in C^{\infty, \infty}([0, T] \times (0, \infty))$.

(ii) For $(t, y) \in [0, T] \times (0, \infty)$ one has
\[
\begin{align*}
\frac{\partial F}{\partial y}(t, y) &= \mathbb{E} \left[ f(y S_t) \frac{W_t}{t^*} \right], \\
\frac{\partial^2 F}{\partial y^2}(t, y) &= \mathbb{E} \left[ f(y S_t) \left( \frac{W_t^2}{t^2} - \frac{W_t}{t^*} - \frac{1}{t^*} \right) \right].
\end{align*}
\]

**Proof.** (a) Let $1 < q, r < \infty$ be such that \( \frac{1}{p} = \frac{1}{q} + \frac{1}{r} \) and \( h(x) := f(e^{x - \frac{x^2}{2}}) \) for \( x \in \mathbb{R} \). Then
\[
\begin{align*}
\int_{\mathbb{R}} e^{-\beta x^2} h(x) dx &= \int_{\mathbb{R}} \left[ e^{-\beta x^2} f(e^{x - \frac{x^2}{2}}) \right] dx \\
&\leq \left( \int_{\mathbb{R}} e^{-\beta x^2} dx \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}} f(e^{x - \frac{x^2}{2}})^q dx \right)^{\frac{1}{p}} \\
&= \left( \int_{\mathbb{R}} e^{-\beta x^2} dx \right)^{\frac{1}{q}} (2\pi)^{\frac{1}{2p}} \| f(S_T) \|_{L^p} < \infty.
\end{align*}
\]

According to Lemma A.1 the function
\[ \psi(s, x) = \mathbb{E} [h(x + W_s)] = \mathbb{E} \left[ f(e^{x + W_s - \frac{s^2}{2}}) \right] \]
is finite for \( x \in \mathbb{R} \) and \( 0 < s < \frac{1}{2p} = T \) and has partial derivatives of all orders. Because of
\[ \psi \left( T - t, \log y + \frac{t}{2} \right) = F(t, y) \quad \text{for} \quad (t, y) \in [0, T] \times (0, \infty) \]
assertion (i) follows.

(b) To show assertion (ii) we observe that
\[
\frac{\partial F}{\partial y}(t, y) = \frac{1}{y} \frac{\partial \psi}{\partial x}(t^*, x) \bigg|_{x = \log y + \frac{t}{2}} = \frac{1}{y} \mathbb{E} \left[ h \left( \log y + \frac{t}{2} + \sqrt{t^*} W_1 \right) \frac{W_t}{\sqrt{t^*}} \right]
\]
and
\[
\frac{\partial^2 F}{\partial y^2}(t, y) = \frac{1}{y^2} \mathbb{E} \left[ h \left( \log y + \frac{t}{2} + \sqrt{t^*} W_1 \right) \left( \frac{H_2(W_1)}{t^*} - \frac{H_1(W_1)}{\sqrt{t^*}} \right) \right]
\]
so that we are done. \( \square \)
Lemma A.3. The function $H : [0, T] \to [0, \infty]$ given by (2) is continuous.

Proof. Let $u_n \to u$ where $u_n, u \in [0, T]$, $P(u, y) := \frac{v^2}{u} - \frac{w}{u} - \frac{1}{u}$ for $(u, y) \in (0, T) \times (0, \infty)$, and $1 < p < 2 < q < \infty$ with $1 = \frac{1}{p} + \frac{1}{q}$. Then, by Lemma A.2,

$$
\mathbb{E} \sup_{n=1,2,...} H(u_n)^2 \leq \mathbb{E} \sup_{n=1,2,...} \left[ \mathbb{E} \left( f(S_{u_n} \mathcal{S}_{u_n^+}) P(W_{u_n^+}, u_n^+) \right) \right]^{\frac{p}{2}} \frac{\mathbb{E} \left[ P(W_{u_n^+}, u_n^+) \right]}{2}^{\frac{q}{2}}
$$

because of our assumption $\mathbb{E} f(S_T)^2 < \infty$ and since $\sup_n \left[ \mathbb{E} P(W_{u_n^+}, u_n^+) \right]^{\frac{q}{2}} < \infty$. Since we also have that $\lim_n H(u_n) = H(u)$ a.s., assertion (i) follows from dominated convergence theorem.

References


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