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Abstract
In this article we investigate the question whether the highly demanding informative requirements of rational expectations models are necessary to derive equilibria within capital market models. In the analysis agents are only provided with publicly available information such as prices and dividends. Nevertheless, we require that agents should behave like econometricians. Additionally, we skip the assumption of rational expectations models that agents know the implied law of motion of the system. By these assumptions, the stock market can be considered as a Sörgel-Hommes consistent expectations model. In this article, we show the existence of consistent expectations equilibria with myopic agents, where the only CEE is the rational expectations equilibrium. In the simulation part we demonstrate how the steady state CEE can be derived by means of sample autocorrelation learning. Thus, we are able to derive a stock market equilibrium with less demanding requirements, where this equilibrium is equal to the rational expectations equilibrium.

JEL-Classification: D83, D84, G10.
Keywords: Artificial Markets, Consistent Expectations, Learning.
1 Introduction

In rational expectation models all agents know the corresponding equations of the economic model describing the economic system. Therefore rational expectation models are highly demanding since they require perfectly informed agents and high computational skills of the agents to solve their optimization problems. These agents are rational in their decision process – since the optimal strategies are derived from maximizing the agents’ utility functions – and in their learning process – since the agents are able to perform their expectations in a statistically correct way. By interaction of rational agents on the corresponding markets, we derive the rational expectations equilibrium (REE). Therefore, the rational expectations equilibrium is a natural consequence of the rational agent paradigm.

When Muth (1961) introduced the concept of REE, most economic models did not care about expectations. Loosely speaking, most economic models assign a constant expectations scheme to their interacting agents, with the result that agents do not or do not properly react to changes in their economic environment, especially to changes in monetary or fiscal policy. Thus, by the concept of rational expectations, the assumption of foolish agents has been replaced by perfectly rational agents. Keeping these disadvantages of non-rational agent models in mind, the rational expectations concept has become the dominating paradigm in economics and finance. Despite the highly demanding requirements, this equilibrium concept serves a benchmark to describe an economic system, where the agents do not make any systematic mistakes.

However, by the rational expectations assumption, the agents often have to know more than economists trying to describe an economic system. The reason is the restrictive requirements concerning the behavior of the agents, the ability of the agents to solve complicated and often nonlinear systems of equations, and their perfect knowledge of the underlying economic model. In addition Sargent (1993) stated that the rational expectations assumption “... typically imputes to the people inside the model much more knowledge about the system they are operating than is available to the economist or econometrician who is using the model to try to understand their behavior”. Nevertheless, it has often been stated the rational expectations can be learned by adaptive learning rules, or models with adaptive agents have their limit in the rational expectations equilibrium. If this statement is correct, the REE model will serve as a good benchmark for other approaches.
Investigating financial markets, a lot of articles use – or refer – their results to the standard CARA model, which is a REE model (See Brunnermeier (1998)). By using myopic agents maximizing utility – which is exponential – rational agents derive their demand functions. Assuming joint normality of dividends and the signals, the equilibrium price is an affine linear function of the expected dividend. Under these assumptions, this REE is also unique. An equivalent result can be derived by using agents maximizing mean-variance utility. The question arises whether the rational expectations equilibrium can be attained by adaptive agents. By using myopic agents using least squares learning rules Bray (1982), Blume et al. (1982), Blume and Easley (1982), and Schönhöfer (1997) derived conditions, for the system to converge to the rational expectations equilibrium. In Timmermann (1994) the convergence of learning depends crucially on the prior information agents impose on the learning process. Thus, if agents try to learn about the long run dynamics without imposing strong prior information, learning cannot converge to REE. If, however, agents impose strong prior information and impose a unit root on prices and dividends of their model, i.e. agents confine their learning to the short run dynamics of the model, learning may converge to REE. Thus, models with adaptive agents may, but they need not converge to REE. An analytical treatment of the question of learning the REE is provided in Marcet and Sargent (1989). In their article agents use least squares learning to update their beliefs every period resulting in the agents’ perceived law of motion; agents are not fully rational in the sense that they neglect their effect on the actual law of motion. The question arises whether the perceived law of motion and the actual law of motion can converge. In a linear setting, this problem could be solved by applying the stochastic approximation tools to the corresponding learning scheme, where the convergence properties of the learning scheme can be reduced to the convergence properties of an ordinary differential equation (See Ljung et al. (1992)). A general treatment of adaptive learning rules would be very difficult. For least squares learning this problem has been investigated by Marcet and Sargent (1989).

In Timmermann (1996) agents are not fully rational in the sense that they do not know the economic model describing the economy. However, agents behave just like econometricians who do not make any systematic mistakes given their underlying information set. The article investigates to what extent agents are able to learn about the true underlying model. By using US stock market data, Timmermann (1996) raised the question whether – under the assumption of a recursive learning scheme – the empirical findings
of excess volatility and predictability of returns in stock markets can be explained. Especially, if dividends follow an autoregressive process with a drift and a trend, he wanted to know whether agents can learn the true model. The article concludes that agents are able to learn; however, learning generates predictability in stock returns and significantly increases the volatility of stock prices.

Additionally, rational expectations equilibria are supposed to result in perfect foresight equilibria, i.e. by the assumption that agents know the law of motion, the distributions of the variables of the system, etc. the agents are able to predict the equilibrium orbits of the system. However, this need not be the case. By splitting up an economic system into an economic law and a set of forecasting rules, Böhm and Wetzelburger (1997) have shown that perfect foresight equilibria need not be attained. This question depends on the forecasting function and on the complexity of the law of the system.

As an alternative to rational expectations bounded rationality models have been proposed. They are based on behavioral foundations (See Sargent (1993), Arthur et al. (1990)) or discrete choice models (See Brock and Hommes (1997) and Brock and Hommes (1998)). However, all models with non-rational agents allow for many degrees of freedom. For example the process of inference of new information can be modeled by means of linear rules, nonlinear techniques, genetic algorithms, and ad-hoc rules. Every forecast technique can result in different equilibrium behavior, if equilibria of the system already exist. Although some models do well in explaining some stylized facts, some ad-hoc assumptions behind bounded rationality models can hardly be justified, at least from the REE point of view.

A test of rational expectations has been performed by Dwyer et al. (1993). In their article the hypothesis that subjective forecasts are essentially the same as forecasts from the relevant economic theory, i.e. from REE, has been tested by means of an experiment. The experiment resulted in no statistically significant evidence of a systematic bias or incorrect use of available information. However, for about 40% of the forecasts, the deviations from the events are predictable ex post, which does not really support the assumption of perfectly rational agents. Therefore, we claim:

1. Agents should not make systematic mistakes given their information set. This results in the requirement that the estimates of the agents be consistent, given a particular statistical model.
2. The information set should only consist of publicly available data, i.e. in the case of our stock market of prices and dividends.

3. An agent should not know the preference orderings and forecast rules of the other agents.

4. The agents should not know the implied law of motion of the economic system.

The question arises what happens if the agents do not know the law of motion of the economic system and agents are only assigned with publicly available information. However, agents should not make systematic mistakes, by applying statistical tools to plan their economic actions.

In this article we investigate this question be means of a simple capital market, where the agents use linear forecasting rules to perform their predictions. This forecasting rule is fixed within our model. The agents are only equipped by their knowledge of past prices and dividends, i.e. \((p_t, d_t)^{T=1}_{t=0}\). This assumption implies very small informational requirements. The information could also be called minimal or costless, since agents or only provided with publicly available information. Based on this information the agents have to predict future prices and dividends. With these estimates agents place orders on the stock market. Then the market clearing system matches these orders into the equilibrium price. However, by the application of linear statistical techniques, every agent is able to perform estimates which are consisten with the data he observes. Therefore, our market is a model where agents do not make systematic mistakes, despite the fact that the agents do not know the law of motion.

These requirements are fulfilled by the concept of Sorger and Hommes (1998) of consistent expectations (CEE). This concept assumes that agents behave like econometricians but do not know the law of motion of the underlying system. An exact definition of consistent expectations will be provided in section 3. As described by Sorger and Hommes (1998) we assume that the agents use a linear forecast rule to estimate the future dividends and prices of a stock. Therefore, the information requirements compared to rational expectations have been decreased enormously. By knowing the mean and the autocorrelation of prices and dividends the agents are able to perform their predictions, resulting in offers at the stock market. No other information requirements, such as the net-demand functions of all other agents, are
required in this setup. By applying this concept to our stock market, the following questions arise:

1. Do there exist consistent expectations equilibria in our stock market model?

2. Are there other equilibria than the REE equilibrium?

3. Can the CEE be learned by agents using autocorrelation learning?

These questions will be investigated in the following way: In section 2, we describe our artificial stock market and the economic agents. After this myopic agents are described in section 2, where asset demands are derived from maximizing mean-variance utility. Section 3 introduces the concept of consistent expectations. As Sorger and Hommes (1997) and Sorger and Hommes (1998) have demonstrated, CEE models can result in steady state CEE, two cycle CEE, and chaotic CEE, depending on the law of motion $F$. In section 4 we show that our artificial stock market can be viewed as an application of the CEE concept. Moreover, the stock market can be reduced to a deterministic model if the dividend process $(d_t)_{t=0}^T$ follows an independent identically distributed process. In this case, we can express the market price $p_t$ as a function of the expected price of the next period $p_{t+1}^e$:

$$ p_t = F \left( p_{t+1}^e \right) $$

(1)

For other dividend processes, the conditionally expected dividend $d_{t+1}^e$ remains a random variable. In section 4 we investigate the question whether consistent expectations equilibria exist in our artificial market with myopic agents. We prove that in this case the stock market is a linear model and the only CEE is the steady state rational expectations equilibrium. This equilibrium is a perfect foresight equilibrium, i.e. the agents perfectly predict the equilibrium trajectory using their forecast rules. The last question stated above is closely related to convergence results of adaptive learning schemes. In this article we shall provide numerical examples of the convergence of the parameters estimated by the agents to the correct parameters in section 6. We shall see that agents using the sample mean and the first order autocorrelation coefficient can learn the true values of the parameters, and the price orbits converge to the steady state of the system. Moreover, this steady state CEE is also the REE. Thus, both concepts result in the same equilibrium, but the REE can be attained at much lower informational requirements.
2 The Stock Market Model

In this section we provide a brief description of the stock market model. Let us consider agents $i$, $i = 1, \ldots, n$ which are able to invest their wealth $w^i_t$ in a risky asset with price $p_t$ and in a risk-free asset paying interest of $r$ per unit of capital invested. The risk-free asset is our numeraire good. Every period $t = 1, \ldots, T$ the risky asset pays dividend $d_t$ which is stochastic. According to these assumptions the budget constraint of agent $i$ becomes:

$$w^i_t = (1 + r)w^i_{t-1} + (p_t + d_t - (1 + r)p_{t-1}) q^i_{t-1},$$

(2)

where $q^i_t$ is the amount of the risky asset held by agent $i$ in period $t$. Secondly, we assume that our agents maximize mean-variance utility, i.e:

$$\max \left[ E_{i,t}(w^i_{t+1}) - \frac{\xi_i}{2} \text{Var}_{i,t}(w^i_{t+1}) \right],$$

(3)

where $\xi_i$ measures the attitude towards risk of agent $i$, $E_{i,t}(.)$ and $\text{Var}_{i,t}(.)$ are the agents' beliefs about the conditional expectation and the conditional variance of the wealth $w^i_{t+1}$ (See Brock and Hommes (1997)). By this assumption the agents need not perform their conditional expectations on prices and dividends in the mathematically correct way. The maximization of expected utility (3) such that the budget constraint (2) is satisfied results in the following demand for the risky asset in period $t$:

$$q^i_{t+1} = \frac{(p_{t+1} + d_{t+1})^c - p_t(1 + r)}{\xi_i \sigma_t^2},$$

(4)

where $p^c_{t+1}$, $d^c_{t+1}$, and $\sigma^2_t$ are agents' beliefs of the conditional expected price of the next period, the conditional expected dividend, and the estimated variance of the price plus dividend, respectively. In our model the information sets $\mathcal{F}_i^t$ are generated by past prices and dividends, where an agent $i$ is assumed to look back or is only able to store the information of the last $h_i$ periods, i.e. $\mathcal{F}_i^t = \sigma(p_{t-1-h_i}, \ldots, p_{t-1}, d_{t-1-h_i}, \ldots, d_{t-1})$. Nevertheless, by setting $h_i = (t-1)$ the whole history of the price and dividend time series is taken into account, which will be assumed in the ongoing analysis.

In the third step we describe the market clearing mechanism. The market clearing price will be derived from intersecting the horizontal sum of individuals' desired demands $q^i_t$ with the supply of shares, which is fixed in our model at $S$. Thus, our market clearing condition becomes:
\[ p_t^* = \sup \{ p_t : \sum_i q_i^t = S \} , \]  

The supply of the risk asset is fixed for every time \( t \). If the sum of net-demands has more than one point of intersection with the supply \( S \), we take the highest price fulfilling the market equilibrium condition that demand equals supply. Therefore, we take the supremum in relationship (5). Since the asset holdings in the last period had to equal the asset supply \( i.e. \sum_i q_{t-1}^i = S \), we could equivalently express (5) in net-demands, yielding:

\[ p_t^* = \sup \{ p_t : \sum_i q_i^t(p_t) = \sum_i q_{t-1}^i \} = \sup \{ p_t : \sum_i q_i^t(p_t) - \sum_i q_{t-1}^i = 0 \} \].

Using agents' demands (4), the market clearing condition (4), and the linearity of demand functions in \( p_t \) we derive:

\[
\begin{align*}
    p_t &= \{ p : \sup_p \sum_i \frac{(p_{t+1} + d_{t+1})^{c_i} - p(1 + r)}{\zeta \sigma_i^2} = S \} \\
    &= \{ p : \sum_i \frac{(p_{t+1} + d_{t+1})^{c_i} - p(1 + r)}{\zeta \sigma_i^2} = S \} .
\end{align*}
\]

Thus, given the price estimates \( p_{t+1}^{c_i} \), the estimates of dividends \( d_{t+1}^{c_i} \), and the estimated variance of prices and dividends \( \sigma_i^2 \), we are able to solve the linear equation (6) for \( p_t \). This implies that the stock price can be described by an autonomous dynamical system, where the \( H(\cdot) \) maps the agents estimates into the market clearing price \( p_t \), i.e.

\[
p_t = H \left( p_{t+1}^{c_1}, \ldots, p_{t+1}^{c_i}, \ldots, d_{t+1}^{c_i}, \ldots, d_{t+1}^{c_i}, \sigma_1^2, \ldots, \sigma_i^2 \right).
\]

### 3 Consistent Expectations Equilibria

In this section we shall provide a brief description of the concept of consistent expectations equilibria developed by Sorger and Hommes (1998). This concept deviates from the concept of rational expectations due to the fact that the agents do not know the economic model behind their economic environment. We could imagine that within this concept agents behave like econometricians trying to interfere from their information set the behavior of the economic system by using linear forecast rules. However, by using the concept of consistent expectations, the agents do not make any systematic mistakes based on their current information in their decision process. In section 4 we shall apply this concept to our artificial capital market, where we
assume that our agents 'believe' in the following AR(1) model, resulting in
the perceived law of motion of prices:

\[
\begin{pmatrix}
\hat{p}_t^i \\
\hat{d}_t^i
\end{pmatrix} = \begin{pmatrix}
\alpha_{p,i} \\
\alpha_{d,i}
\end{pmatrix} + \begin{pmatrix}
\beta_i & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
\hat{p}_{t-1} - \alpha_{p,i} \\
\hat{d}_{t-1} - \alpha_{d,i}
\end{pmatrix} + \begin{pmatrix}
\varepsilon_{p,t} \\
\varepsilon_{d,t}
\end{pmatrix},
\]

(7)

where \(\varepsilon_{p,t}, \varepsilon_{d,t}\) are assumed to be independent of each other and iid. with 
zero mean and variance \(\sigma_p^2\) and \(\sigma_d^2\) respectively. Therefore, our agents use 
the following model to predict the one period ahead price:

\[\hat{p}_{t+1}^i = \alpha_{p,i} + \beta_i (\hat{p}_{t-1} - \alpha_{p,i}) .\]

(8)

For the dividends, equation (7) results in the following forecast:

\[\hat{d}_{t+1}^i = \alpha_{d,i} .\]

(9)

As already stated in section 2, the agents need the two step ahead estimate 
of prices \((\hat{p}_{t+1}^i)\) and dividends \((\hat{d}_{t+1}^i)\) in their demand functions (4). After 
recursive substitution this yields:

\[\hat{p}_{t+1}^i = \alpha_{p,i} + \beta_i^2 (\hat{p}_{t-1} - \alpha_{p,i}) =: l_i(\hat{p}_{t-1}) ,\]

(10)

and

\[\hat{d}_{t+1}^i = \alpha_{d,i} .\]

(11)

**Remark 1** The reader should note that at this stage of analysis we have not 
determined how the parameters are estimated by our agents. In section 5 the 
parameters \(\alpha_p, \beta, \) and \(\alpha_d\) will be estimated by means of sample autocorrelation 
learning. The estimates at period \(T\) will be given by \(\hat{\alpha}_{p,t}, \beta_t, \hat{\alpha}_{d,t}\) based on 
past prices and dividends \((p_t, d_t)_{t=0}^{T-1} .\)

By the agents’ assumption of independence of prices and dividends, the conditional variance of prices and dividends \(\sigma^2_i\) will be derived form the following equation:

\[\sigma^2_i = \sigma^2_{p,i} + \sigma^2_{d,i} .\]

(12)

Since our agents do not know the market clearing mechanism and the de-
mands of other agents, the affine linear functions \(\hat{p}_{t+1}^i = \alpha_i + \beta_i (\hat{p}_{t-1} - \alpha_i)\) are 
the perceived laws of motion of our agents, while the composition of demand
functions (4), the forecasts of the agents, and the market clearing mechanism $H(.)$ results in the implied law of motion of our capital market. The sequence of prices $(p_t)_{t=0}^{T}$ has a sample average $\bar{p}_t$ and an autocorrelation coefficient $\gamma_{j,t}$ of order $j$. The asymptotic sample average is defined by:

$$\bar{p} = \lim_{t \to \infty} \frac{1}{t-1} \sum_{i=0}^{t-1} p_i .$$

(13)

The autocorrelation coefficients for every lag $j$ are given by:

$$\gamma_j = \lim_{t \to \infty} \frac{c_{j,t}}{c_{0,t}} ,$$

where $c_{j,t}$ is derived from:

$$c_{j,t} := \frac{1}{t-j} \sum_{i=0}^{t-j} (p_i - \bar{p}_{t-j}) (p_{t+j} - \bar{p}_t) .$$

(15)

Considering the law of motion $F$, which has been derived from the composition of the market clearing mechanism and the AR(1) forecast rules, Sorger and Hommes (1998) have defined a consistent expectations equilibrium (CEE) by:

**Definition 1** A consistent expectations equilibrium is a triple $\{(p_t)_{t=0}^{\infty}; \alpha, \beta\}$, where $\alpha$ and $\beta$ are the parameters of the AR(1) process used as the forecast rule for $p_t$, such that $p_{t+1} = \alpha p_t + \beta (p_{t-1} - \alpha p_t)$. This triple fulfills the following conditions:

1. The sequence $(p_t)_{t=0}^{\infty}$ fulfills the implied law of motion $F$.
2. The sample average of the prices $\bar{p}_t$ is equal to $\alpha_p \forall t$, while the sample average of dividends prices $\bar{d}_t$ is equal to $\alpha_d \forall t$.
3. For sample autocorrelation coefficients $\gamma_{t,j}$ the following is true: If $(p_t)_{t=0}^{\infty}$ is a convergent sequence, then $sgn(\gamma_j) = sgn(\beta^j)$, $j \geq 1$, and $\gamma_j = \beta^j$, for lags $j \geq 1$ for a divergent $(p_t)_{t=0}^{\infty}$.

Thus, by the CEE concept, the requirement that agents know the law of motion of the economic system has been dropped. The CEE concept results in a set of equilibria including the rational expectations equilibrium but allows for other behavior in equilibrium that has been excluded by the stringent assumption of rational expectations. As shown by Sorger and Hommes (1997)
and Sorger and Hommes (1998), CEE can result in steady state CEE, two-cycle CEE and chaotic CEE.

**Remark 2** According to Sorger and Hommes (1998) the agents do not check whether the conditions of definition 1 are fulfilled. The agents simply use their forecast rules to predict the future, where the parameters have already been derived by a particular learning algorithm. Therefore, the parameters are consistent with the agents’ estimates given the forecast rule. The forecast rule will not be changed if the forecast error is high. The question whether sample autocorrelation learning yields fulfills the requirements of definition 1 in a simple capital market model will be considered in section 6.

### 4 The Stock Market in the CEE Framework

Now we would like to show that our artificial market is a model of type M2 if the dividends \((d_t)_{t=0}^\infty\) follow an independent identically distributed process with finite mean and variance, \(\bar{d}\) and \(\sigma_d^2\), respectively. Hommes and Sorger have defined a model of type M2 by a \(x_t = G(x_{t+1})\), where \(G\) is a deterministic map.

From our considerations in section 2 and section 3 we already know that the stock market can be described by the map \(H(,)\) and the agents’ forecasts of prices, dividends, and variances. However, from the definition of the CEE we know that the estimates \(p_{t+1}^e, d_{t+1}^e,\) and \(\sigma^2\) have to be equal for all agents in equilibrium. Therefore, the market price becomes a function \(p_t = \hat{H}(p_{t+1}^e, d_{t+1}^e, \sigma^2)\). Since the dividend process is a stochastic process the market price \(p_t\) still remains a random variable. However, if we assume that the dividends follow in independent identically distributed process with mean \(\alpha_d\) and variance \(\sigma_d^2\), \(d_{t+1}^e\) has to be equal to the constant \(\alpha_d\) in a consistent expectations equilibrium. Therefore, \(p_t = \hat{H}(p_{t+1}^e, \alpha_d, \sigma^2)\). By these considerations and our market clearing condition (6) we derive:

\[
p_t = \left\{ p : \sum_i p_{t+1}^e + \alpha_d - p(1 + r) = S \right\} . \quad (16)
\]

Inserting the forecast-rule (10) into (16), and a few algebraic manipulations yields:
\begin{align*}
p_t &= \frac{1}{1 + r} \left( p_{t+1}^c + \alpha_d - \frac{S \sigma^2}{\sum_i \xi_i} \right) \\
&= \frac{1}{1 + r} \left( \alpha_p + \beta^2 (p_{t-1} - \alpha_p) + \alpha_d - \frac{S \sigma^2}{\sum_i \xi_i} \right). \quad (17)
\end{align*}

Therefore, by equation (17), we have derived an explicit expression for the law of motion of prices. Since \( \alpha_d \) and \( \sigma_d^2 \) are constants, equation (17) defines a deterministic mapping \( F(p_{t-1}; \alpha_d, \alpha_p, \beta, \sigma^2) \). Therefore, we obtain a model of type M2 if a CEE exists. This has not been checked for our capital market model. Therefore, we will apply the following lemmas of Sorger and Hommes (1997) to equation (17). The existence of a CEE will be shown by calculating the steady state CEE of the capital market model.

**Lemma 1** If \( \{ (p_t)_{t=0}^{\infty}, \alpha, \beta \} \) is a steady state CEE converging to \( p^* \), then \( \alpha_p = p^* \), and \( p^* \) is a fixed point of the law of motion \( F \).

**Proof** If \( \lim_{t \to \infty} p_t \) exists, the sample average of dividends and prices converges to its limit by the law of large numbers. This implies that \( \alpha_p = p^* \) in equation (8), satisfying the second requirement of definition 1. Considering the law of motion as described in (17), a steady state equilibrium implies a fixed point. \( \Box \)

After this we would like to know whether the reverse holds, resulting in the following lemma:

**Lemma 2** Assume \( \hat{p} \) to be a fixed point of \( F(p_{t-1}, \alpha_d, \sigma^2) \), i.e., \( p_t = \hat{p} \) for all \( t \), with \( \alpha_p = \hat{p} \) and \( \beta \in [-1, 1] \). Then \( \{ (p_t)_{t=0}^{\infty}, \alpha, \beta \} \) is a steady state CEE with \( \hat{p} = p^* \).

**Proof** From the fixed point property of the mapping \( F \) we derive a sequence of prices \( \hat{p}, \hat{p}, \ldots \), with \( \alpha_p = \hat{p} \). Now consider definition 1: The sequence of prices follows the actual law of motion \( F \), with expected \( p_t \) equal to \( \alpha_p = \hat{p} = p^* \) implying a steady state. The second condition in the definition of consistent expectations equilibria is met due to \( \alpha_p = \hat{p} \). The autocorrelation coefficient \( \beta \) is still in the interval \([-1, 1]\). \( \Box \)
**Remark 3** The reader should note that in a steady state CEE the variance $\sigma^2$ is equal to $\sigma^2_d$. Since the price is constant at the fixed point of $F$, the variance $\sigma^2_p$ is equal to zero. Only the dividends remain stochastic, causing the variance $\sigma^2_d$.

Therefore, from equation (17), the fact that $\sigma^2$ is equal to $\sigma^2_d$, and $p_{t+1} = \alpha_p$ in a steady state CEE where $p_{t+1} = \alpha_p = p_t$, the mapping $F$ becomes a linear function in $p_{t-1}$. Therefore, we derive:

$$p_t = \frac{1}{1 + r} \left[ \left( \alpha_d - \frac{S\sigma^2_d}{\sum_i \frac{1}{\zeta_i}} \right) + p_{t+1}' \right],$$

(18)

where the intercept of (18) is equal to:

$$\frac{1}{1 + r} \left( \alpha_d - \frac{S\sigma^2_d}{\sum_i \frac{1}{\zeta_i}} \right),$$

(19)

and the slope is given by $\frac{1}{1 + r}$. Due to a strictly positive interest rate $r$ the slope of the function is positive and less than one. Now, let us investigate the question of a fixed point of $\tilde{H}(p_{t+1}', \alpha_d, \sigma^2_d)$ in the first quadrant. Since the slope of (18) is positive and less than one the intercept (19) of the function (18) has to be positive. This requires that $\alpha_d \geq \frac{S\sigma^2_d}{\sum_i \frac{1}{\zeta_i}}$. From an economic point of view this implies that a nonnegative share price requires an average dividend per share $\alpha_d$ higher than the product of the number of shares times the variance $\sigma^2_d$, divided by the sum of the inverses of the coefficients of absolute risk aversion. After we have derived a necessary condition for positive prices for the risky asset, we derive the fixed point $p_t = p_{t+1} = p^* = \alpha_p$ from (18):

$$p^* = \frac{\frac{1}{1 + r} \left( \alpha_d - \frac{S\sigma^2_d}{\sum_i \frac{1}{\zeta_i}} \right)}{1 - \frac{1}{1 + r}}$$

$$= \frac{\alpha_d - \frac{S\sigma^2_d}{\sum_i \frac{1}{\zeta_i}}}{r}. $$

(20)

Considering (20) the reader can easily check that: $\frac{\partial p^*}{\partial \alpha_d} > 0$, $\frac{\partial p^*}{\partial r} < 0$, $\frac{\partial p^*}{\partial \sigma^2_d} < 0$, and $\frac{\partial p^*}{\partial \kappa_i} < 0$. Additionally (20) states that the steady state price is equal to
with is the risk neutral value of an asset minus a term taking care for the fact that the agents are risk averse. In Appendix A we show that the steady state CEE is the only possible CEE in our capital market model.

**Remark 4** The reader can easily verify that the CEE given by (20) is also the REE. However this steady state equilibrium is attained by less demanding requirements as the REE.

**Remark 5** Since the price trajectory is equal to the constant $p^*$ in equilibrium, and the forecast (8) of the agents $p_t^*$ is equal to $p^*$, the CEE is a perfect foresight equilibrium.

In the last step we want to check the stability of our steady state CEE. Therefore, let us consider our system (17). To derive to stability results we have to calculate the eigenvalues of the Jacobian matrix of $F(p_{t+1}) = F(l(p_{t-1}), \alpha, \sigma)$, where $l(p_{t-1})$ is our AR(1) prediction provided by (8) with parameters $\alpha$, $\beta$ converging for all agents in equilibrium. Because we have a one-dimensional system, we only have to calculate the first derivative of $F(.)$. By applying the chain rule to $F$ and the result that $\sigma^2_p$ is a constant in equilibrium, we derive:

$$\frac{d}{dp_{t-1}} F(p_{t-1}, \alpha, \sigma^2)(p^*) = \frac{\beta^2}{1 + r} + \frac{2S}{(1 + r) \sum_i \sigma^2_i} \frac{d \sigma^2_p}{dp_{t-1}} = \frac{\beta^2}{1 + r}. \quad (21)$$

Considering equation (21) the first derivative of $F$ is nonnegative. To derive stability for our system (17) the magnitude of the first derivative has to be less than one. This condition is fulfilled iff $\frac{1}{1 + r} \beta^2 < 1$ or $\beta \leq \sqrt{1 + r}$. Thus, we have shown that a CEE exists in our artificial market and the only CEE is the steady state rational expectations equilibrium.

## 5 Sample Autocorrelation Learning

Generally, our stock market allows different learning schemes to estimate the parameters of the forecast model (7), resulting in different predictions of the future prices and dividends. For example the agents use sample autocorrelation learning, least squares learning, or agents use another adaptive learning
rule, where histories are allowed to be of different length \( h_i \). By using sample autocorrelation learning the agents simply have to calculate the sample averages of the price \( \bar{p}_{t,i} \)

\[
\bar{p}_{t,i} = \frac{1}{t-1} \sum_{i=0}^{t-1} p_t ,
\]  

and the first order autocovariances:

\[
COV_{t,i}(p_t, p_{t-1}) := \frac{1}{t-1} \sum_{i=1}^{t-1} (p_t - \bar{p}_{t,i}) (p_{t-1} - \bar{p}_{t-1,i}) .
\]

The sample average and the autocovariance of dividends are calculated in the same way. The variance of prices is derived by calculating \( COV_{t,i}(p_t, p_t) \) in equation (23), while the coefficient \( \beta \) is derived from:

\[
\gamma_{t,i} := \frac{COV_{t,i}(p_t, p_{t-1})}{\sqrt{COV_{t,i}(p_t, p_t)} \sqrt{COV_{t,i}(p_{t-1}, p_{t-1})}} .
\]

Therefore, we have derived estimates for \( \alpha_p, \alpha_d, \beta, \) and \( \sigma \) by means of sample autocorrelation learning. These estimates can now be used in the forecast models (8) and (9) to describe the agents’ learning behavior by means of simulations.

6 Simulations

In this section we investigate the question whether the CEE derived in section 4 can be learned by means of sample autocorrelation learning in a simulated stock market. If this would be the case, the rational expectations equilibrium can be attained by less demanding assumptions concerning the agents information set and their knowledge of the law of motion. Nevertheless, we have to keep in mind that our simulations neither provide a proof that sample autocorrelation learning results in convergence to the steady state nor serve conditions of convergence.

Figure 1 presents one particular time series for 500 trading periods with 10 agents using sample autocorrelation learning. Heterogeneity enters via the degree of risk aversion \( \zeta_i \). In the following simulations \( \zeta_i \) is uniformly distributed on the interval \([0.05, 0.15]\). The initial value of prices \( (p_0) \) and dividends \( (d_0) \) are exogenously determined. The price \( p_0 \) where no trade
Figure 1: Price Time Series

takes place was generated by \( p_0 = p^* + \varepsilon_{0,p} \) with \( \varepsilon_{0,p} \sim N(0, 100) \) and the rational expectations equilibrium \( p^* \). For the dividends we have \( d_0 = \alpha_d + \varepsilon_{0,d} \) with \( \varepsilon_{0,d} \sim N(0, \sigma_d^2) \). This pre-trade data is necessary to perform parameter estimates. The fixed interest rate was set to \( r = 0.1 \) per time step, the dividend mean to \( \alpha_d = 10 \), the supply of stock to \( S = 1 \), and the variance of dividends to \( \sigma_d^2 = 4 \).

The yield \( \rho_t = (p_t + d_t - p_{t-1})/p_{t-1} \) remains above the fixed interest yield most of the time. In our simulations we performed 200 runs with different initial values and different \( \zeta_i \) randomly drawn from the interval [0.05, 0.15]. Table 1 presents the mean yield, the variance of yields, the skewness, and the kurtosis of the simulated time series. The terms in parentheses in the second row present the variance of these statistics. To compare our results to empirical data we used daily DAX-yields from July, 19th 1996 to October, 2nd 1998 and daily S&P 500 yields, from January 1996 to August 1997. Since we have not calibrated the simulation parameters to the empirical data, only the skewness and the kurtosis should be used for comparisons. The kurtosis is of our main interest, since we would like to know whether our market model explains the fat tails property observed in empirical data (See Campell et al. (1997)).
In a next step let us consider the autocorrelation functions. As shown in table 2 the autocorrelation of yields is statically insignificant for both the DAX- and the S&P yield time series respectively. The last two columns are the BOX-Pierce Q-statistic and the Ljung-Box Q-statistic (See Campbell et al. (1997)). Both statistics perform a test on null-hypothesis that all autocorrelations are zero against the alternative that they are significantly different from zero. The critical values are taken from a $\chi^2$ distribution, where the degrees of freedom are equal to the number of autocorrelation coefficients estimated. In our simulations and the empirical series the critical value is approximately equal to 8.12. The autocorrelation coefficients of yields in the empirical time series are not significant, as well as in 79.5% of the simulation runs. Once again the terms in parentheses are the variances of the corresponding statistics. In the empirical time series we observe autocorrelations in squared yields. This effect is often called volatility clustering. In our simulations this effect is only significant in 21.8% of the simulation runs. In the mean volatility clustering cannot be observed in our simulated price time series. Additionally, volatility clustering was only significant when the autocorrelation of yields was significant as well. Therefore, we are not able
to derive volatility clustering in these simulated yield series.

Last but not least we want to check whether the parameters $\alpha_p$, $\alpha_d$, and $\beta$ fulfill or approximately fulfill the CEE requirements of definition 1. According to equation (19) the REE is equal to $p^* = 99.98$. After 500 time steps, the mean $\alpha_p = \bar{p}_{500}$ is equal to 99.7002, with a variance of 35.63. Since the mean of prices is normal by the central limit theorem we are able to perform a test with the null-hypothesis $H_0 : p^* = \alpha_p$ against the alternative $H_1 : p^* \neq \alpha_p$ at a significance level of 95%. The reader can easily check that in the above case $H_0$ cannot be rejected. In the case of dividends we derive a $d_{500,i}$ of 10.0065 with an estimation variance of 0.0074. Also with the estimate $\tilde{d}_{500,i}$ is not significantly different from $\alpha_d = 10$. In the last step we have to check whether $\beta$ is equal to the autocovariances of prices. Since $\beta$ is the autocovariance coefficient of the prices in the case of sample autocorrelation learning, $\beta$ and $\gamma_1$ have to agree. However, according to theorem 9 in Sorger and Hommes (1997), $\beta$ has to be nonnegative since $\tilde{H}$ is increasing in $p_{t+1}$. In our simulation runs the mean is equal to 0.6699 with a variance of 0.1415 after 500 time steps. Nevertheless, we have in 6.36% of our estimates a negative $\beta$ in our estimates. However, this percentage is slightly decreasing since we have a percentage of 9.54% and a percentage 8.19% of negative $\beta$s after 300 and 400 time steps, respectively. Last but not least, the mean of the prices is equal to 99.8134 with a variance of 28.7960 after 500 time steps. This implies that we have approximately reached the consistent expectations equilibrium in almost all runs after 500 time steps. However, it should be noted that convergence cannot be proved by means of simulation runs. This question has to be subject to further research.

7 Conclusions

In this article we considered an artificial stock market as an application of the concept of consistent expectations equilibria. In this framework we skip the highly demanding requirements of perfect information and the knowledge of the law of motion of the economic system used in rational expectations models. In the approach presented here, the agents do not make any systematic mistakes, based on their forecast model. However, they neither know the law of motion nor the preferences of the other agents. We have shown analytically that by assuming linear forecast rules, providing only publicly available information to each agent, assuming an independent iden-
tically distributed dividend process, and using myopic agents the consistent expectations equilibrium is equal to the rational expectations equilibrium. In the simulation part we obtained the result that the steady state CEE is approximately derived after 300 to 500 time steps in almost all of 200 simulation runs. However, we should keep in mind that we cannot prove any convergence results by means of numerical tools. Nevertheless the CEE can be approximately attained by sample autocorrelation learning in our capital market model. Thus the REE can – in many cases – be learned by agents who simply calculate mean values and autocorrelation coefficients from publicly available information. Therefore the REE can be attained with less demanding requirements.

However, there are a lot of questions remaining. According to the definition of consistent expectations equilibria, the agents have already learned the parameters of the system by using their prediction models. Therefore we need analytical results whether the CEE can be learned by means of sample autocorrelation learning. Secondly, the reason for non-linearities in empirical time series of share prices cannot be solved by this dynamic system where the system is linear and prices converge to a steady state. We have to check in which ways nonlinear prediction rules can create non-linearities, and whether nonlinear prediction rules are plausible from the empirical point of view.

Although the CEE capital market model is not able to explain the stylized facts stated above, we derived the result that the REE of this capital market model can also be obtained by a weaker equilibrium concept, where agents do not know the actual law of motion of the system and the stochastic characteristics of the dividend process. Therefore, we derived the same equilibrium with less informed agents, since the agents simply receive past prices and dividends to perform their forecasts.
A Uniqueness of the CEE

After we have shown the existence of a steady state CEE in section 4, we want to investigate whether there are any other CEE in our capital market model. First of all let us restrict our analysis to bounded CEE, i.e. \( 0 \leq (p_t)_{t=0}^\infty \leq K < \infty \). This is also a reasonable restriction from the economic point of view, since all agents have a limited budget and prices far above some level – for given \( r \) and \( (d_t)^T_{t=0} \) – are implausible for the agents. Additionally, for every \( \alpha_d < \infty \) and \( \sigma_d^2 < \infty \), such a bound can be derived from equation (17). Since we do not require that \( \sigma_p^2 \) converges in the definition of the CEE (Definition 1), let us check the estimated variance of prices \( \hat{\sigma}_{p,t}^2 \). The question arises whether there exist other CEE with nonzero variance of prices \( \sigma_p^2 \).

The sample variance of prices \( \hat{\sigma}_{p,t}^2 \) remains nonlinear in \( p_t \) in this dynamic system. The sample variance of dividends is a consistent estimate converging to \( \sigma_d^2 \) by the law of large numbers. Secondly, the sample variance of prices is derived from:

\[
\hat{\sigma}_{p,t}^2 = \frac{1}{t-1} \sum_{i=0}^{t-1} p_i^2 - \left( \frac{1}{t-1} \sum_{i=0}^{t-1} p_i \right)^2 . \tag{25}
\]

This is a deterministic sequence \( \left( \hat{\sigma}_{p,t}^2 \right) \) since \( \hat{H} \) is deterministic map, for a given \( \alpha_d \) and \( \sigma_d^2 \). For \( \hat{\sigma}_{p,t}^2 \) to be finite the sum of squared prices has to converge. Since \( p_t \) is bounded by the function \( K \), it is easy to see that \( \hat{\sigma}_{p,t}^2 < K^2 < \infty \) for all \( t \). Next, let us verify that \( \left( \hat{\sigma}_{p,t}^2 \right) \) is a Cauchy sequence, and therefore convergent to a constant \( \bar{\sigma}_p^2 \). By the definition of the CEE (See Definition 1), the mean of prices is equal to \( \alpha_p \). Thus,

\[
|\hat{\sigma}_{p,t}^2 - \hat{\sigma}_{p,t-1}^2| = \left| \frac{t-2}{(t-1)(t-2)} \left( \sum_{i=0}^{t-2} p_i^2 + p_{t-1}^2 \right) - \frac{1}{t-2} \sum_{i=0}^{t-2} p_i^2 \right|
\leq \frac{p_{t-1}^2}{(t-1)} \leq \frac{K^2}{(t-1)} = K^2 \varepsilon , \tag{26}
\]

for \( (t-1) \geq N(\varepsilon) \). Next, for every constant \( \bar{\sigma}_p^2 \), the mapping \( F \) is linear and monotone increasing in \( p_{t-1} \) for positive values of \( \beta \). Since \( \sigma_p^2 \) has to converge to a constant in equilibrium, \( p_t = F(p_{t-1}, \alpha_d, \alpha_p, \sigma_p^2) \) has to be a linear system in a consistent expectations equilibrium. This mapping can result in behaviors as described in Guckenheimer and Holmes (1993). Nevertheless, we can
apply theorem 9 of Sorger and Hommes (1997) to prove our results. This theorem states that for a model of type M2, a map $\hat{H}$ increasing in $p_{r+1}$ results in a steady state CEE with a nonnegative autocorrelation coefficient, while a decreasing $\hat{H}$ can either result in steady state CEE or in a two-cycle CEE.

Now let us match this theorem with our system (17). As already stated, the function $F$ is monotone and increasing, resulting in a unique steady state equilibrium, where $\sigma_p^2 = \hat{\sigma}_p^2$ has to be equal to zero. As stated in Sorger and Hommes (1997), the coefficient $\beta$ has to be nonnegative. From the economic point of view this implies a positive first order autocorrelation of prices. This property is also commonly observed in empirical data on share prices (See Campell et al. (1997)).
References


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